k-regular self-similar fragmentation process

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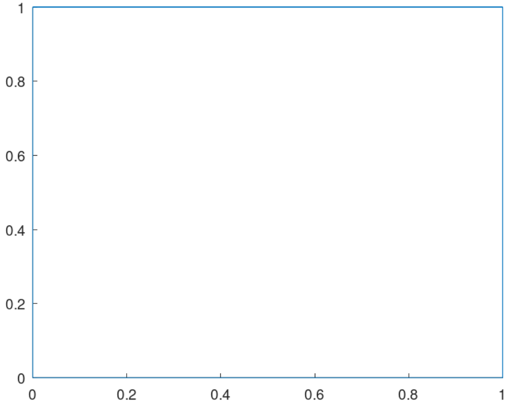
P. D., Nina Gantert, Samuel G.G. Johnston, Joscha Prochno, and Dominik Schmid. „Sharp concentration for the largest and smallest fragment in a k-regular self-similar fragmentation" The Annals of Probability 50, no. 3 (2022): 1173-1203.



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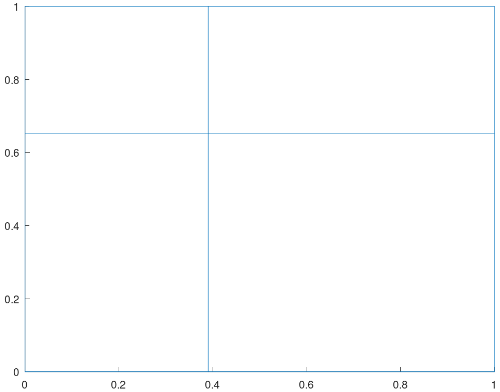
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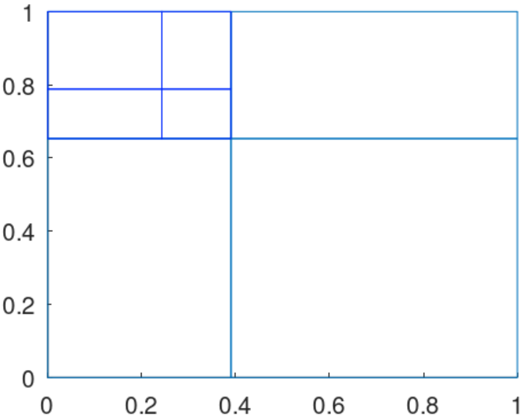
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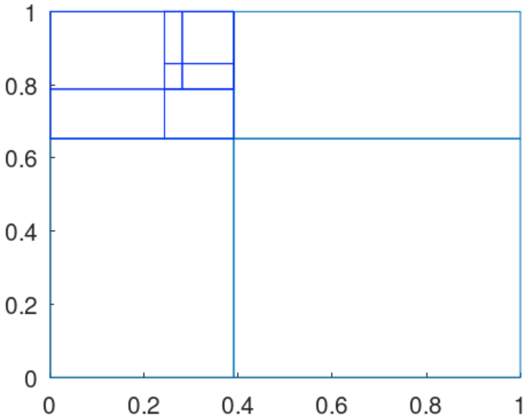
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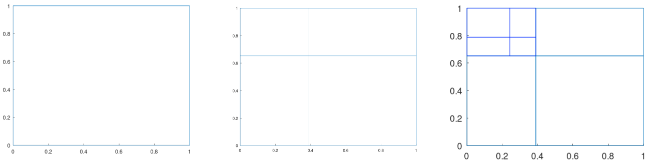
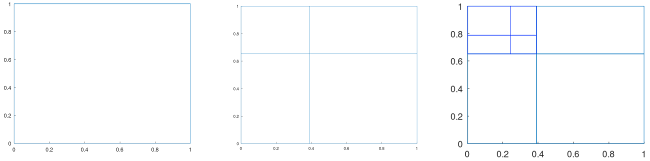


Self-similar fragmentation is a Markov process $(I_t)_{t\geq 0}$ on

$$\mathcal{S} = \left\{ (u_1, u_2, u_3, \ldots) : u_1 \geq u_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} u_i \leq 1 \right\}.$$



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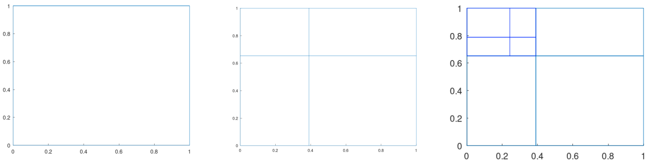

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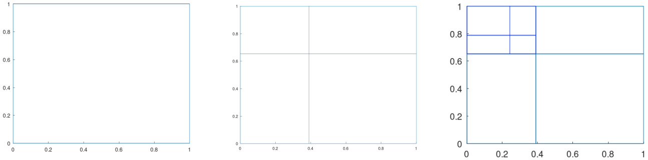

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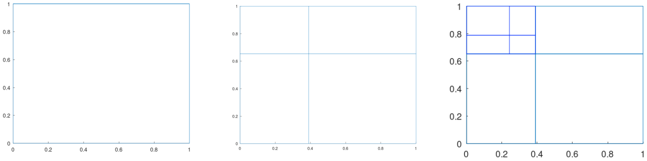

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- each fragment of size u has exponential lifetime with parameter $u^{-\alpha}$
- ν - dislocation measure $\mathcal{S}$ (probabilistic)
- each fragment of size u splits into fragment of sizes $uT_1 \geq uT_2 \geq uT_3 \dots$, with $(T_1, T_2, \dots) \in \mathcal{S}$ is distributed according to ν

Theorem ([Filippov, 1961], [Bertoin, 2004])

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Let $\alpha \geq 0$, $I_t = (X_i(t))_{i\geq 1}$. If $\kappa(p) = 1$ for some $p > 0$, then

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$$\mathcal{M}_\infty = \lim_{t \to \infty} \mathcal{M}(t).$$

Theorem ([Kolmogorov, 1941], [Biggins, 1990])

Let $\alpha = 0$. For any continuous and bounded $f\colon \mathbb{R} \to \mathbb{R}$,

$$\sum_{k=1}^{\infty} X_k^p(t) f\left(\frac{1}{t}\log(X_k(t))\right) \to^{\mathbb{P}} f(\kappa'(p))\mathcal{M}_\infty.$$

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$$\mathbb{P}[X^*(t) = X_k(t) | I_t] = \frac{X_k(t)^p}{\mathcal{M}(t)}, \qquad \mathcal{M}(t) = \sum_{i=1}^{\infty} X_i(t)^p$$

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Let $\alpha = 0$.

$$\frac{1}{t}\log(X^*(t)) \to^{\mathbb{P}} \kappa'(p).$$

$$\frac{\log X^*(t) - t\kappa'(p)}{\sqrt{t\kappa''(p)}} \to^{d} \mathcal{N}(0,1).$$

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Let $\alpha > 0$.

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Theorem ([Kyprianou, Lane and Mörters, 2017])

$\alpha = 0$,

$$-\log(M_t) = c_1 t + c_2 \log(t)(1 + o(1))$$

Theorem ([Bertoin, 2003])

Let $\alpha > 0$.

$$-\log(M_t) = \frac{1}{\alpha} \log(t)(1 + o(1)).$$

Definition

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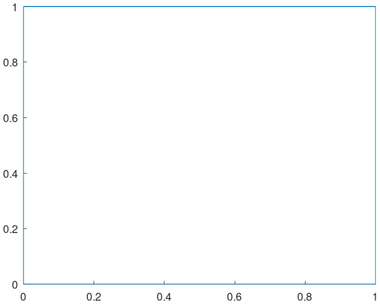

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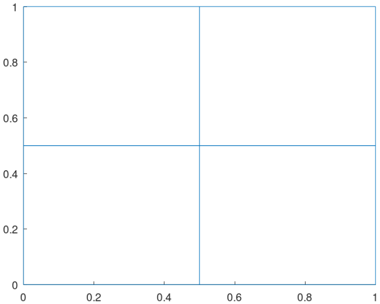

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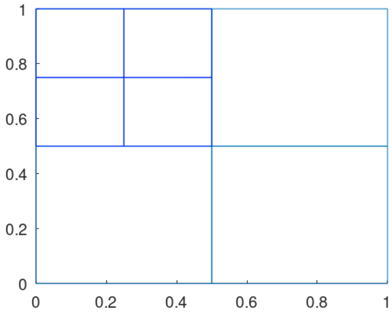

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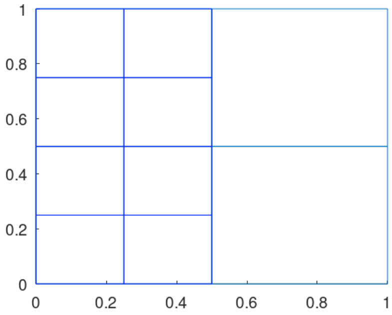

M_t - size of the largest fragment, m_t - size of the smallest fragment Let

$$h(t) = \frac{1}{\alpha}\log_k(t) - c_1\log_k\log_k(t) - c_2,$$
$$g(t) = \frac{1}{\alpha}\log_k(t) + c_3\sqrt{\log_k(t)} - c_4\log_k\log_k(t) + c_5.$$

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Theorem (D, Gantert, Johnston, Prochno, Schmid, 2022)

For k-regular fragmentation process such that $\alpha > 0$ and $I_0 = (1, 0, 0, \ldots)$, with probability one for sufficiently large $t > 0$,

$$\left\lceil h(t) - d_1 \frac{\log\log(t)}{\log(t)} \right\rceil \le -\log_k(M_t) \le \left\lceil h(t) + d_1 \frac{\log\log(t)}{\log(t)} \right\rceil,$$
$$\left\lceil g(t) - d_2 \frac{1}{\log(t)^{1/3}} \right\rceil \le -\log_k(m_t) \le \left\lceil g(t) + d_2 \frac{1}{\log(t)^{1/3}} \right\rceil$$

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Remark

If $\alpha < 0$,

$$S_n \to \sum_{i=0}^{\infty} k^{\alpha i} W_i < \infty$$

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$$t^{\frac{1}{\alpha}} k^{-Y_t} \to R^{\frac{1}{\alpha}}$$

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$[0, 1)$

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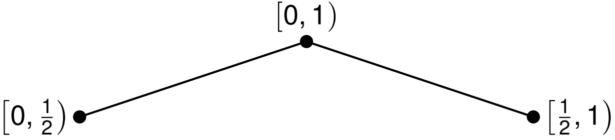

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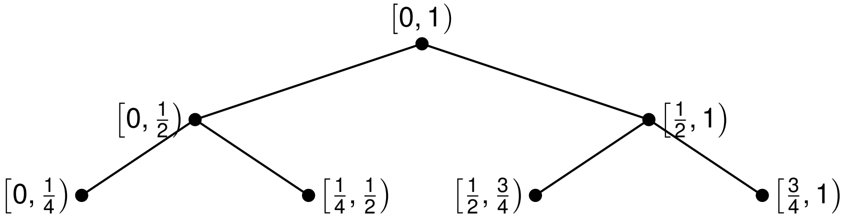

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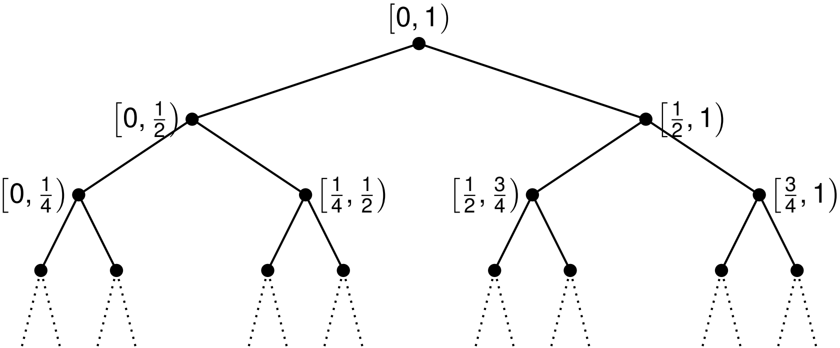

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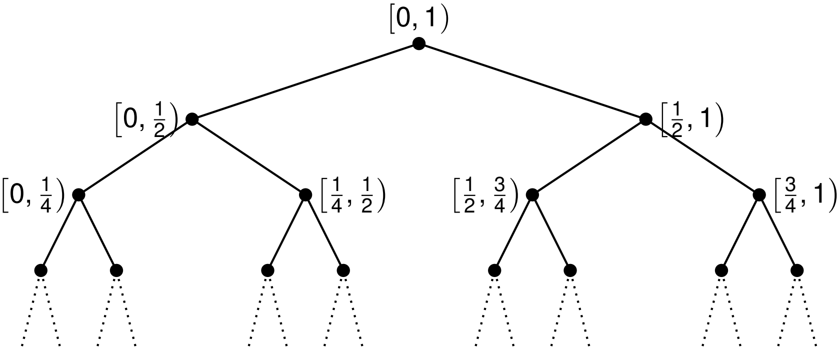

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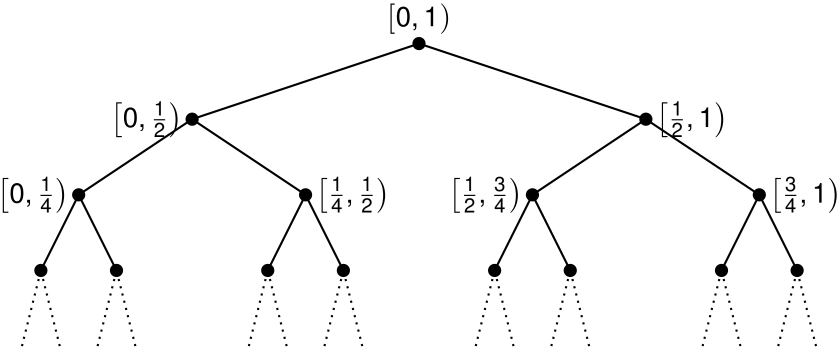

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(Ex. $S([0, k^{-n})) = S_n$).
If $v \subseteq w$ i $|v| = |w| + 1$,

$$S(v) = S(w) + k^{\alpha |v|} W^{(v)}$$

$(S(v))_{v \in \mathbb{T}}$ is so-called expanding branching random walk [Athreya, 1985]

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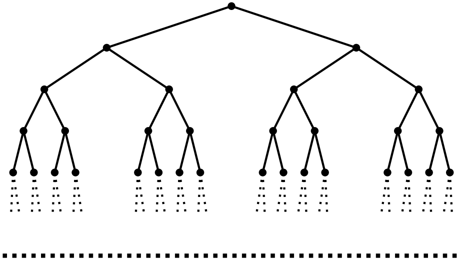

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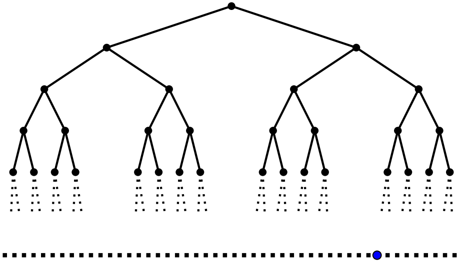

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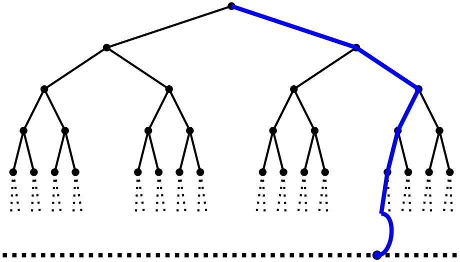

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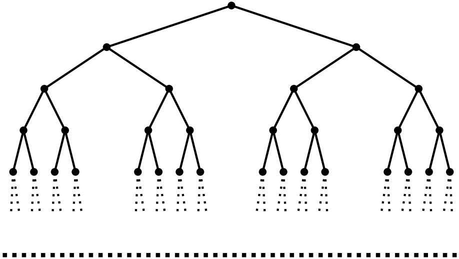

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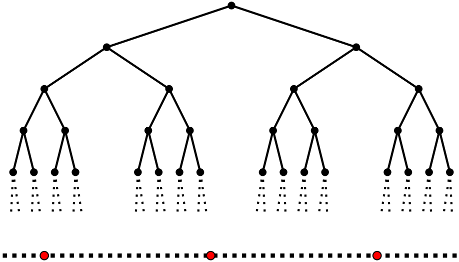

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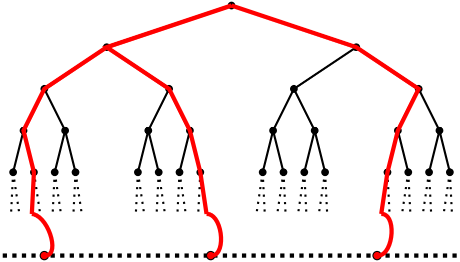

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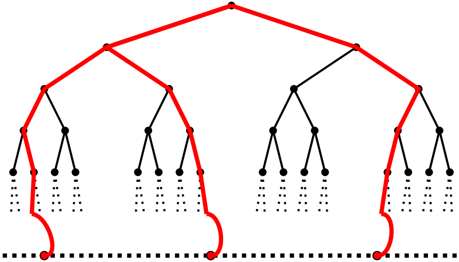
$$S(v) = \sum_{y \leq v} k^{\alpha|y|} W^{(y)}, \qquad \max_{|v|=n} k^{\alpha|v|} W^{(v)}$$



$$k^{-\alpha n} \sup_{|v|=n} S(v) \approx \sup_{v \in \ldots} R(v)$$

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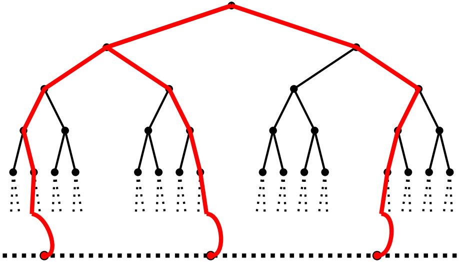

$$k^{-\alpha n} \sup_{|v|=n} S(v) \approx \sup_{v \in \ldots} R(v)$$

$$k^{-\alpha n} \sup_{|v|=n} S(v) - n \log(k) \to^{d} F(x) = e^{-c_R e^{-x}}$$

M_t - size of the largest fragment, m_t - size of the smallest fragment Let

$$h(t) = \frac{1}{\alpha}\log_k(t) - c_1 \log_k \log_k(t) - c_2,$$
$$g(t) = \frac{1}{\alpha}\log_k(t) + c_3\sqrt{\log_k(t)} - c_4 \log_k \log_k(t) + c_5.$$

for some known constants $c_i > 0$.

Theorem (D, Gantert, Johnston, Prochno, Schmid, 2022)

For k-regular fragmentation process such that $\alpha > 0$ and $l_0 = (1, 0, 0, \ldots)$, with probability one for sufficiently large $t > 0$,

$$\left\lceil h(t) - d_1 \frac{\log\log(t)}{\log(t)} \right\rceil \le -\log_k(M_t) \le \left\lceil h(t) + d_1 \frac{\log\log(t)}{\log(t)} \right\rceil,$$
$$\left\lceil g(t) - d_2 \frac{1}{\log(t)^{1/3}} \right\rceil \le -\log_k(m_t) \le \left\lceil g(t) + d_2 \frac{1}{\log(t)^{1/3}} \right\rceil$$

for some constants $d_1, d_2 > 0$.

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