The largest fragment in self-similar fragmentation processes

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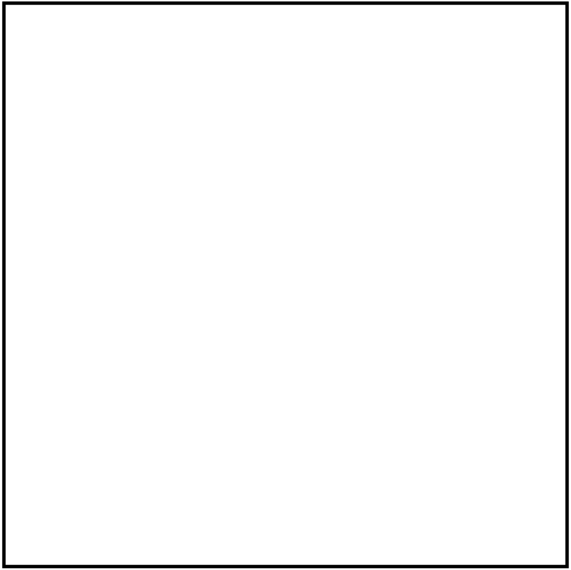
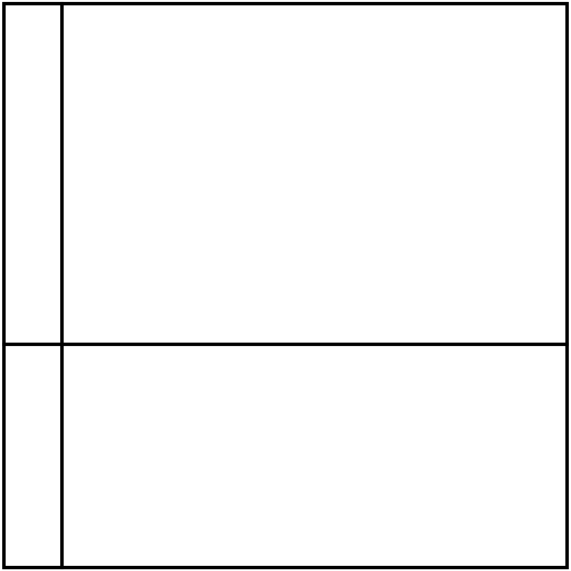
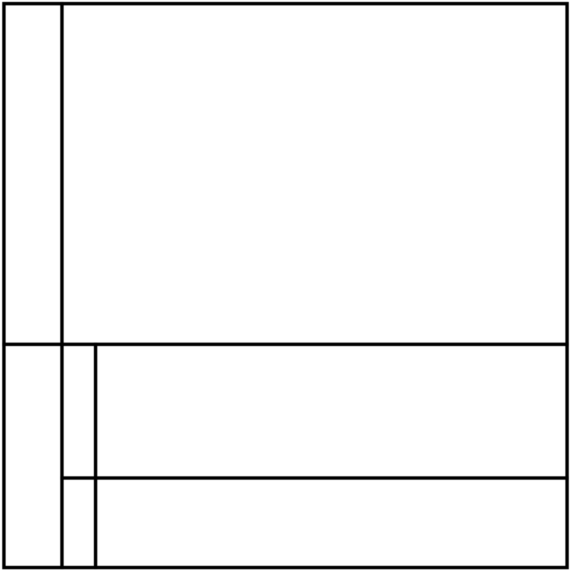
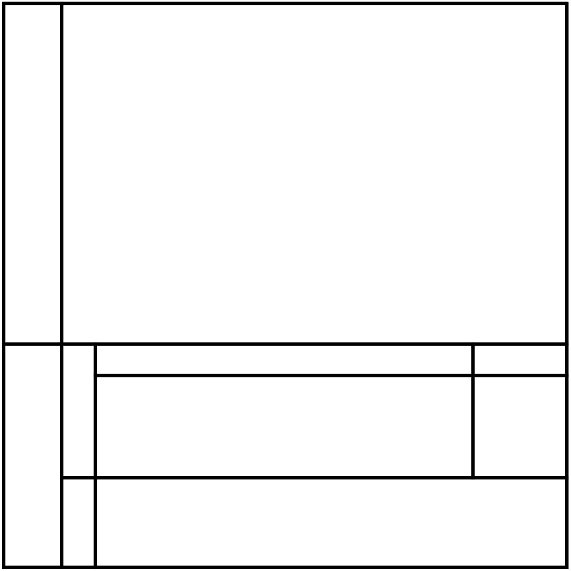
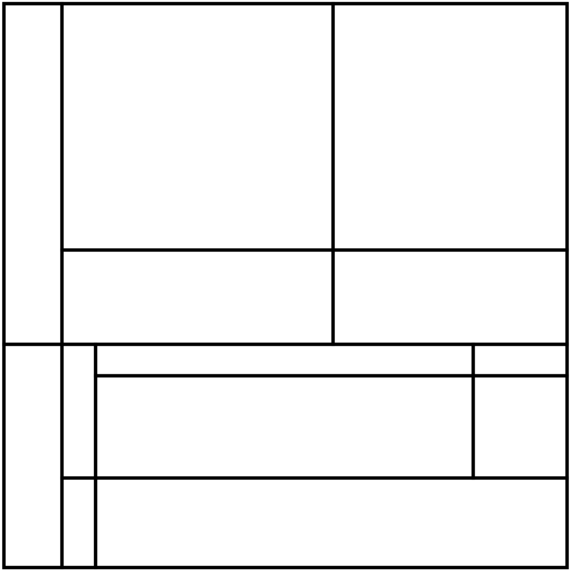
PD, Samuel Johnston, Sandra Palau, Joscha Prochno, *The largest fragment in self-similar fragmentation processes of positive index* arXiv preprint arXiv:2409.11795 (2024).

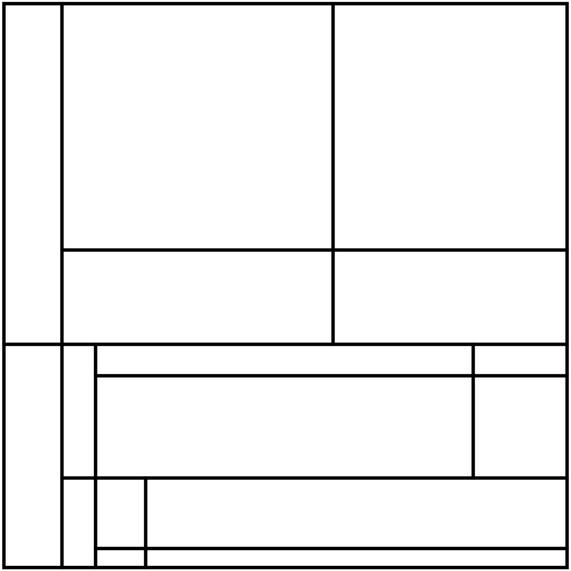


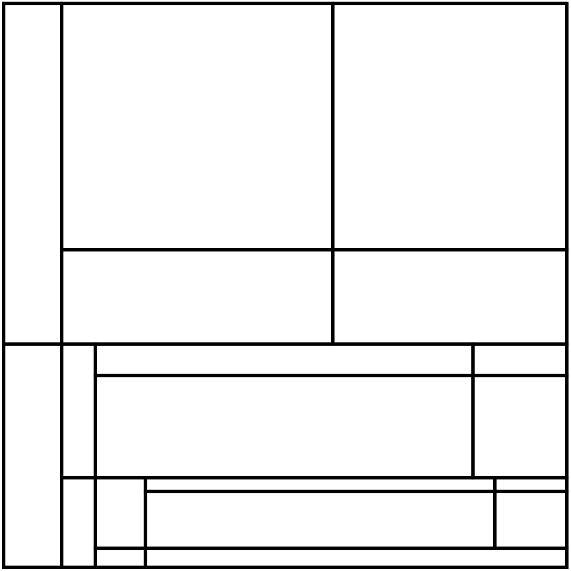


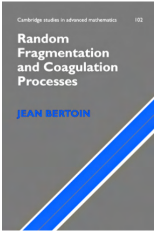








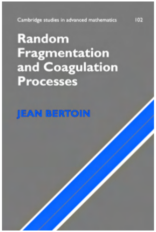




[Bertoin, PTRF 2001] [Bertoin, AIHP 2002]
[Bertoin, JEMS 2003]

A self-similar fragmentation is a Markov process $I_t = (X_1(t), X_2(t), \ldots)$ on

$$\mathcal{S} = \left\{ \mathbf{s} = (s_1, s_2, s_3, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{k \in \mathbb{N}} s_k \leq 1 \right\}.$$



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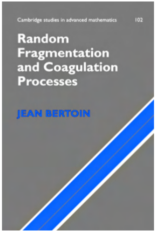

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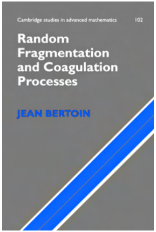

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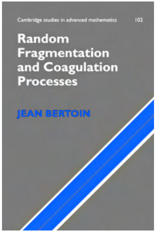

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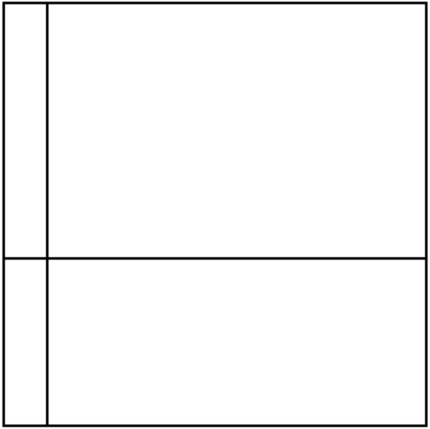
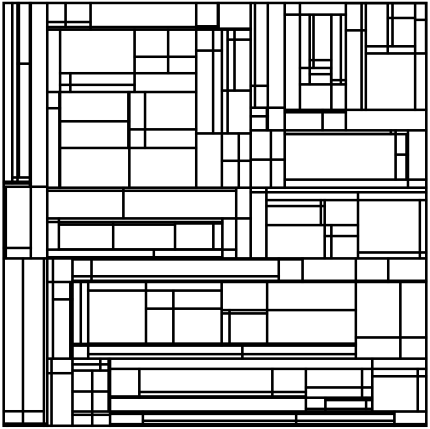
- $\alpha \in \mathbb{R}$ the index of self-similarity;
- fragment of size u has an exponentially distributed lifetime with parameter $\nu(\mathcal{S})u^{\alpha}$;
- each fragment of size u splits into fragments of sizes $uS_1 \geq uS_2 \geq uS_3 \ldots$, where $(S_1, S_2, \ldots)$ is sampled according to $\nu(\cdot)/\nu(\mathcal{S})$;



$U, V \sim \mathcal{U}[0,1]$,

$$(S_1, S_2, \dots) = (UV, U(1-V), (1-U)V, (1-U)(1-V), 0, \dots)^{\downarrow}$$



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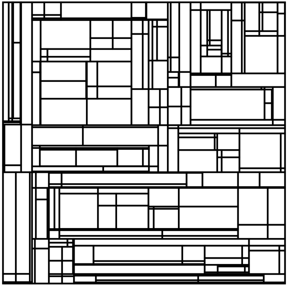
$U^* \sim \mathcal{U}[0,1]^2$
$\chi(t)$ = size of the rectangle containing U^* at time t.



$I_t = (X_1(t), X_2(t), \dots) \in \mathcal{S}$

$$\chi(t) | I_t \stackrel{d}{=} \sum_{k \in \mathbb{N}} X_k(t) \delta_{X_k(t)}$$

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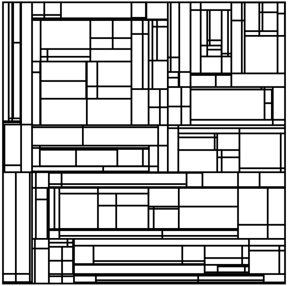

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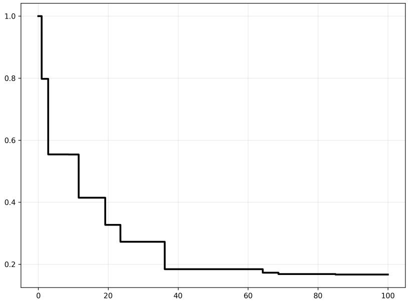
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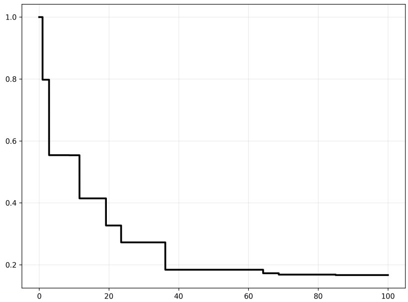
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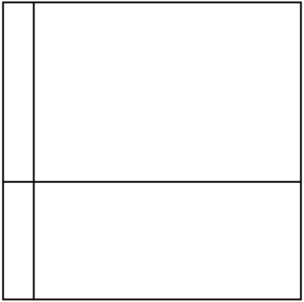

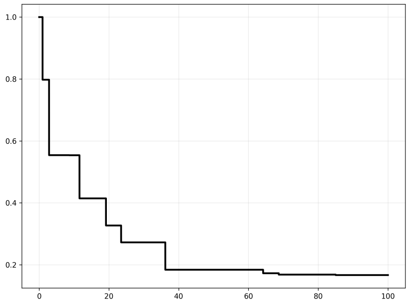
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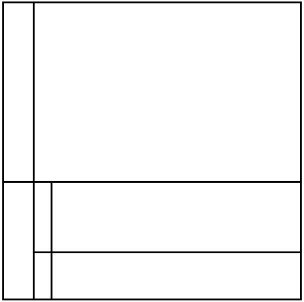

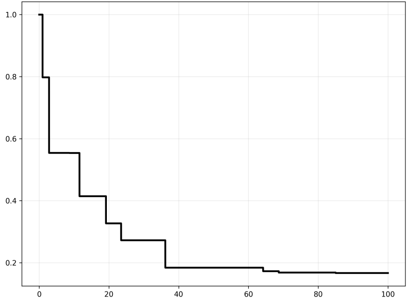
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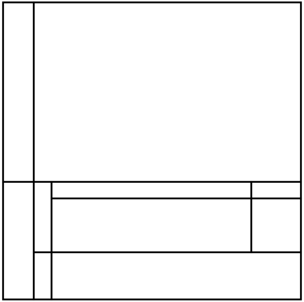

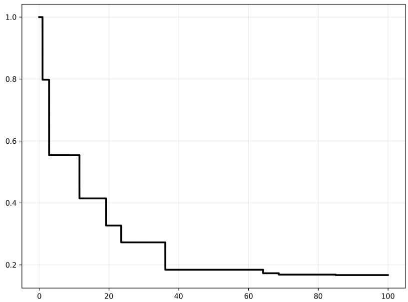
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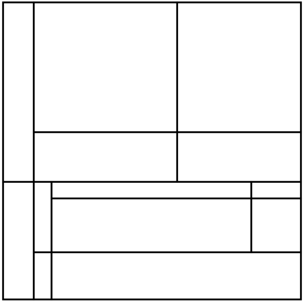

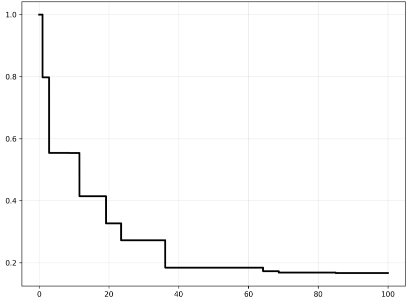
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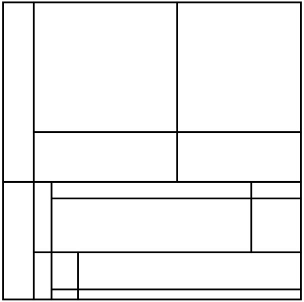

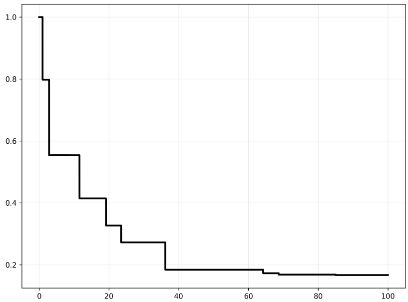
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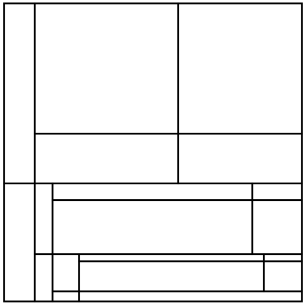

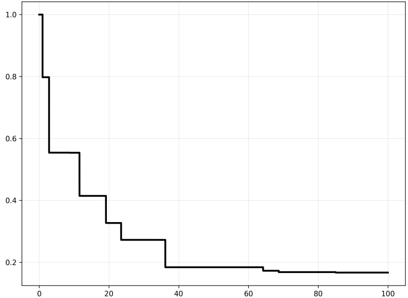

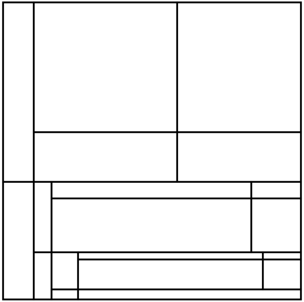

$$L_\chi f(x) = \nu(\mathcal{S}) x^\alpha \mathbb{E}\left[\sum_{k\in\mathbb{N}} S_k f(S_k x) - f(x)\right]$$
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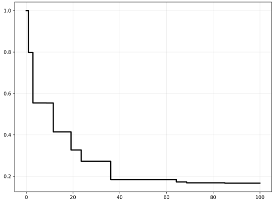
$$\left(\chi^{(x)}(t)\right)_{t\geq 0} \stackrel{d}{=} \left(x\chi^{(1)}(x^{\alpha}t)\right)_{t\geq 0}$$



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there exist a Lévy process ξ

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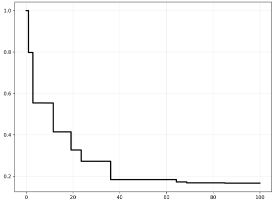

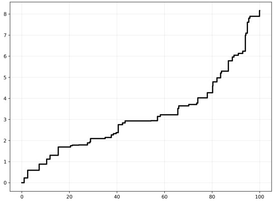

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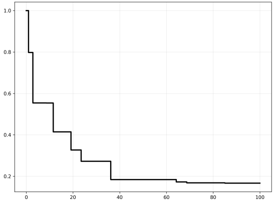

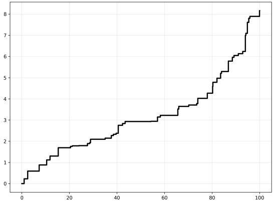

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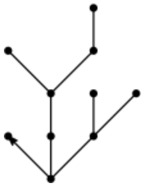
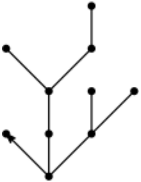
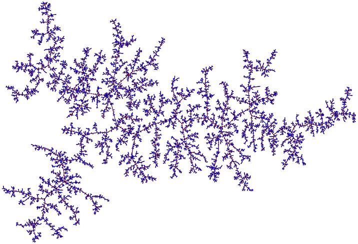
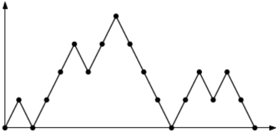
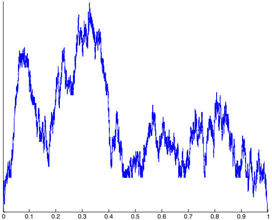
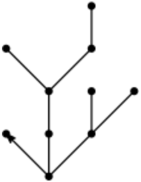
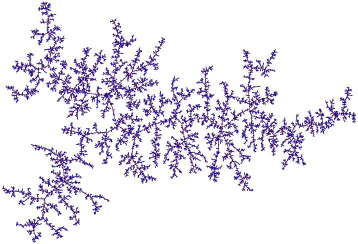


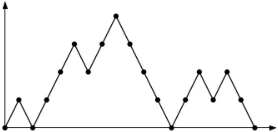
[Aldous & Pitman, AoP 1998]

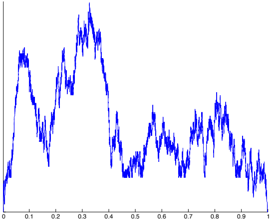





[Aldous & Pitman, AoP 1998]









[Aldous & Pitman, AoP 1998]

$$\int_{\mathcal{S}} g(s_1, s_2, \ldots)\nu(\mathrm{d}\mathbf{s}) = \frac{1}{2}\int_0^1 g(u, 1-u, 0, \ldots)\frac{1}{\sqrt{2\pi u^3(1-u)^3}}\mathrm{d}u$$

ν can be infinite provided that

$$\int_{\mathcal{S}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty, \qquad \mathbf{s} = (s_1, s_2, \ldots).$$

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$\chi(t) = e^{-\xi_{\rho(t)}}$

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$$L_{\xi} f(x) = \int_{\mathcal{S}} \left(\sum_{k \in \mathbb{N}} s_k f(\log(1/s_k) + x) - f(x) \right) \nu(\mathrm{d}\mathbf{s})$$

$$\mu = \mathbb{E}[\xi_1] = \int_{\mathcal{S}} \sum_{k \in \mathbb{N}} s_k \log(1/s_k) \nu(\mathrm{d}\mathbf{s}) < \infty, \qquad \xi_t \sim \mu t$$

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$$\int_0^{\rho(t)} e^{\alpha \xi_r} \mathrm{d}r = t, \qquad \rho(t) \sim \frac{1}{\alpha\mu} \log(t)$$

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Theorem (D, Gantert, Johnston, Prochno, Schmid, AoP 2022)

$$\nu = \delta_{(1/k, 1/k, \ldots, 1/k, 0, \ldots)}, \quad X_1(t) \sim c t^{-1/\alpha} (\log(t))^{1/\alpha}$$

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$$\leq \mathbb{E}\left[\sum_{k \in \mathbb{N}} \mathbf{1}_{\{X_k(t) > e^{-h}\}}\right] = \mathbb{E}\left[\chi(t)^{-1} \mathbf{1}_{\{\chi(t) > e^{-h}\}}\right] \leq e^h \mathbb{P}\left[\xi_{\rho(t)} \leq h\right]$$

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$$\mathbb{P}\left[\xi_t \leq \epsilon\right] \approx \exp\left\{-t^{\frac{1}{1-\theta}}\epsilon^{-\frac{\theta}{1-\theta}}\ell^{-1}(t/\epsilon)\right\}$$

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$$\Phi(q) = \int_{\mathcal{S}}\left(1 - \sum_{k\in\mathbb{N}} s_k^{q+1}\right)\nu(\mathrm{d}\mathbf{s}) \sim Cq^{\theta}\ell(q)$$

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$$\approx \exp\left\{-\left(e^{-\alpha h}t\right)^{\frac{1}{1-\theta}} \ell^{-1}\left(e^{-\alpha h}t\right) \inf_{\int f = 1} \int \left(f(s)e^{s}\right)^{\frac{1}{1-\theta}} \mathrm{d}s\right\}$$