Large deviations for a random walk in random environment revisited

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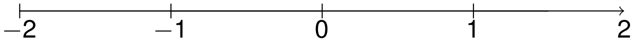
(joint with Dariusz Buraczewski)

Będlewo, 7.05.2018

$\omega = \{p_k\}_{k\in\mathbb{Z}}$, $p_k \in (0,1)$ are iid.



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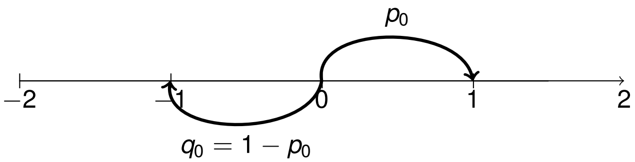

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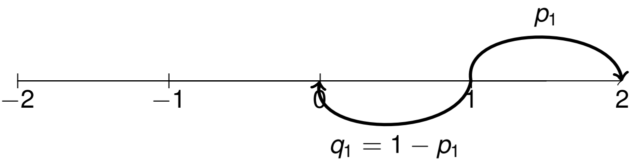

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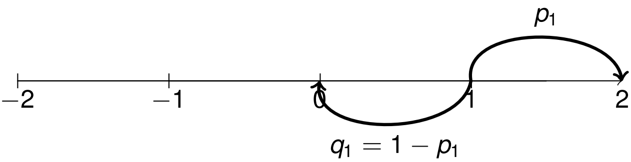

$X = \{X_n\}_{n\in\mathbb{N}}$ is a random walk in random environment (RWRE)

$$P_\omega[X_{n+1} = k+1 \,|X_n = k] = p_k$$
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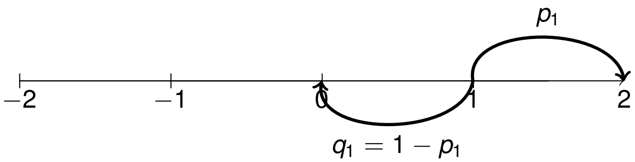

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P_ω – quenched probability.

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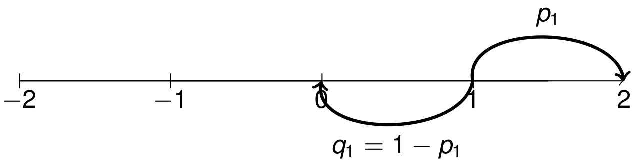

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P_ω – quenched probability. Define the annealed probability $\mathbb{P}$ viz.

$$\mathbb{P}[X \in A,\, \omega \in B] = \int_B P_\omega[X \in A]\, P(\mathrm{d}\omega).$$

Let $A = \frac{q}{p}$. Assume

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Solomon (Ann. Probab. '75)

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Kesten, Kozlov, Spitzer (Compos. Math. '75)

$$\alpha \in (1,2), \quad \frac{X_n - vn}{n^{\frac{1}{\alpha}}} \Rightarrow \mathcal{L}_{\alpha}, \qquad \alpha > 2, \quad \frac{X_n - vn}{n^{\frac{1}{2}}} \Rightarrow \mathcal{N}$$

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Buraczewski, D. (submitted '17)

$$\mathbb{P}[X_n < (v-\varepsilon)n] \sim C\varepsilon^{-\alpha} n^{1-\alpha}$$

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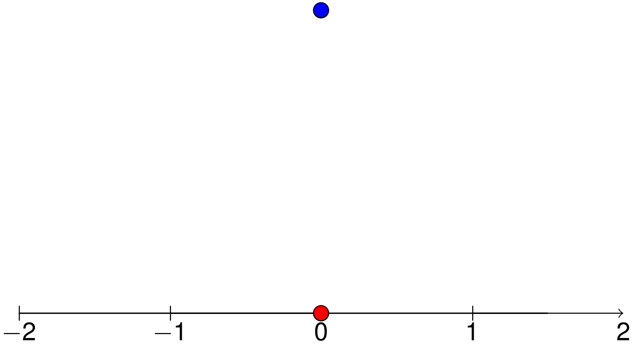
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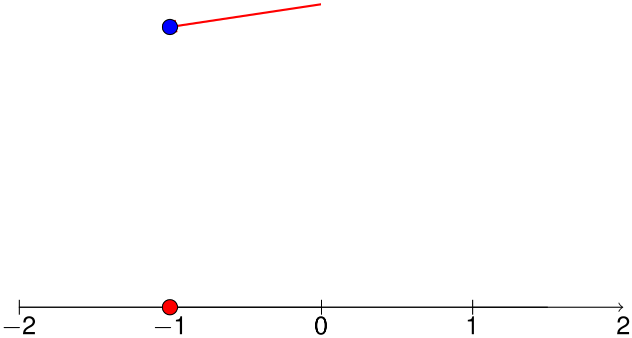
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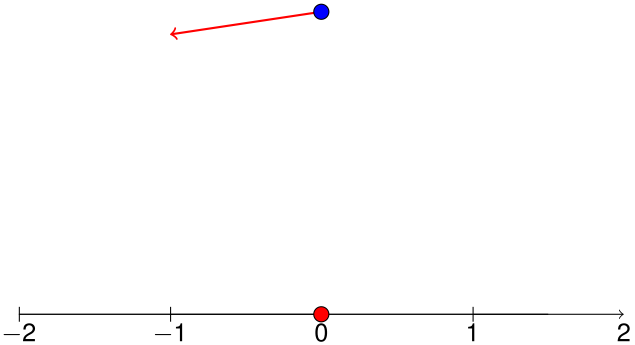
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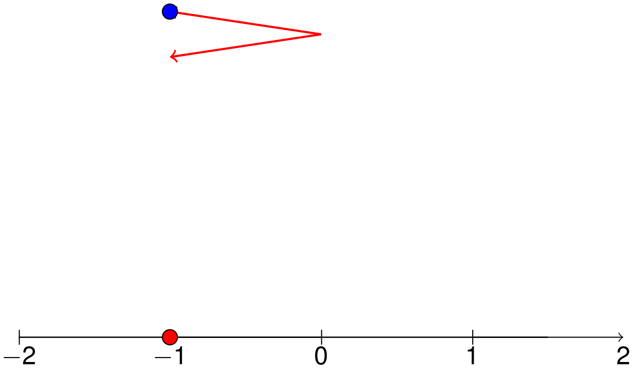
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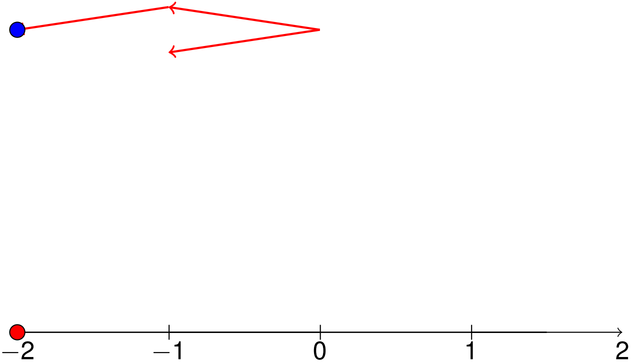

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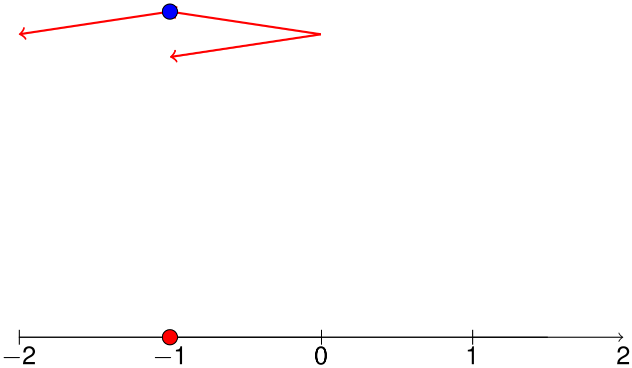
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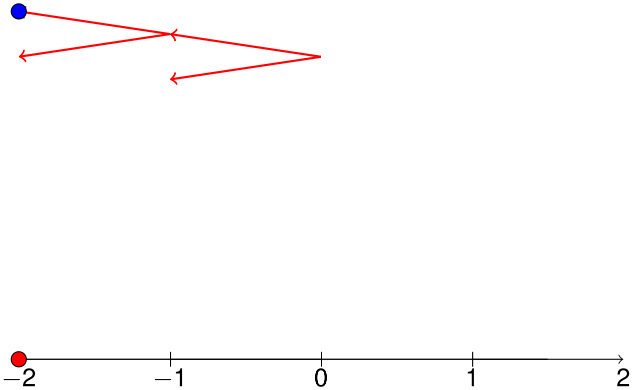
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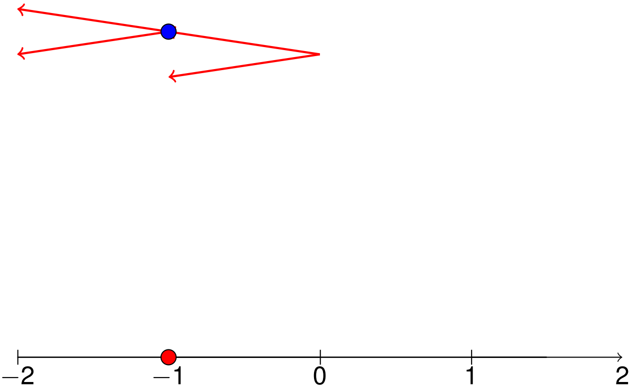
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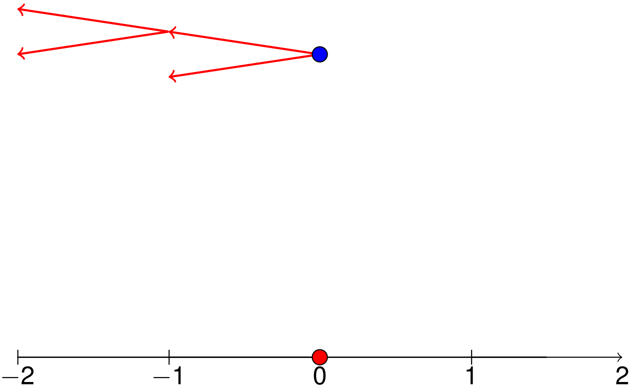
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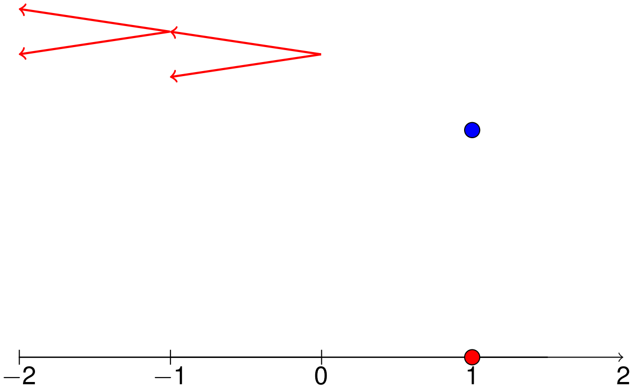
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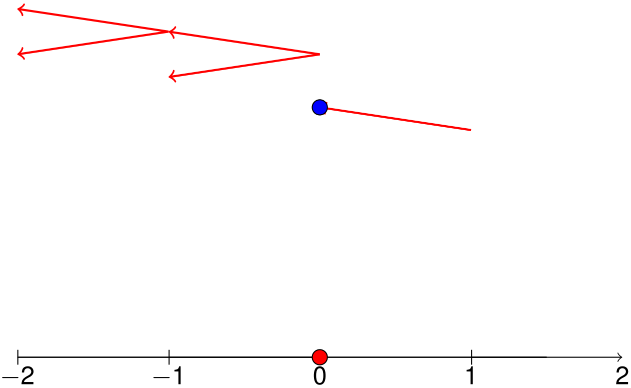

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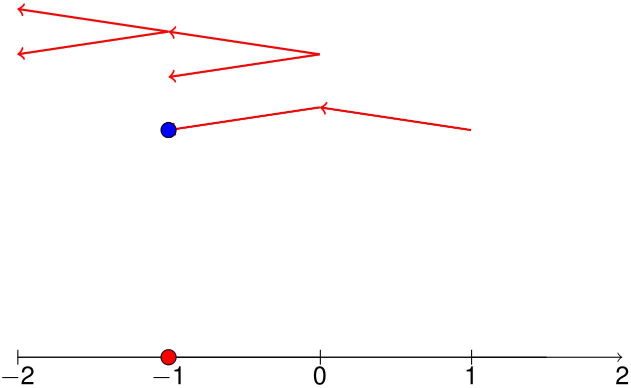
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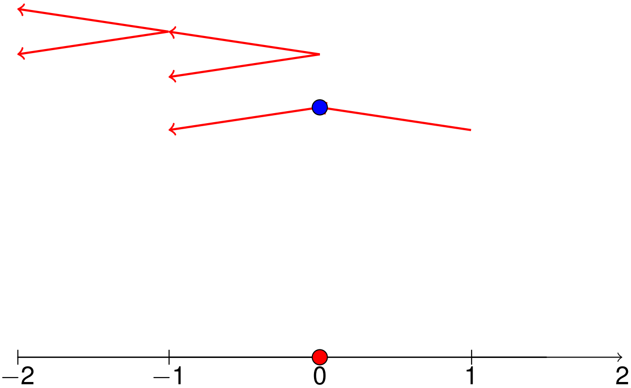
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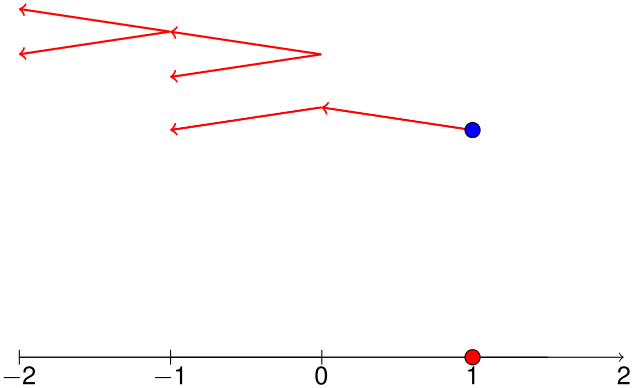

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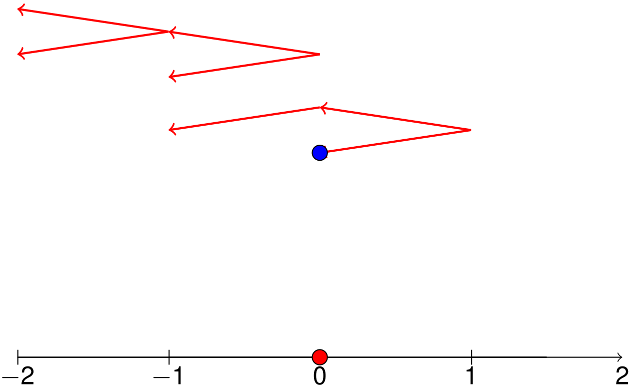

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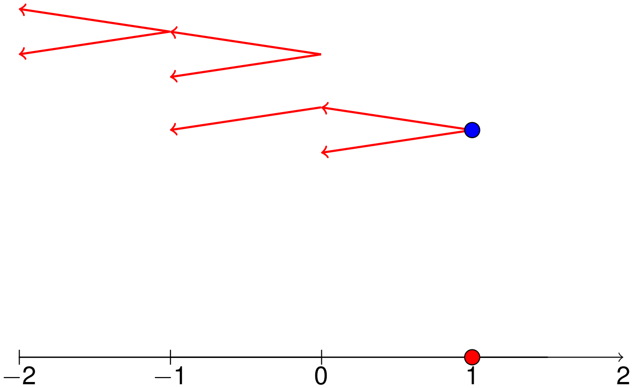
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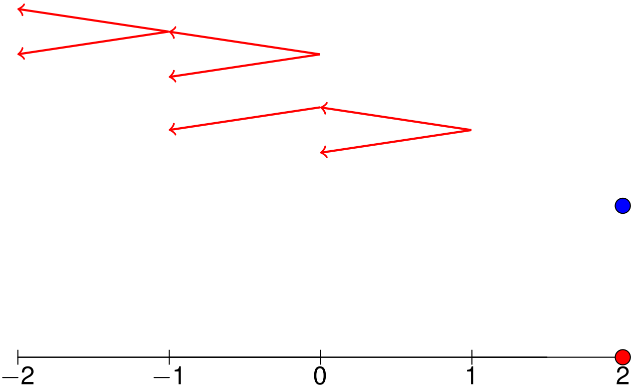
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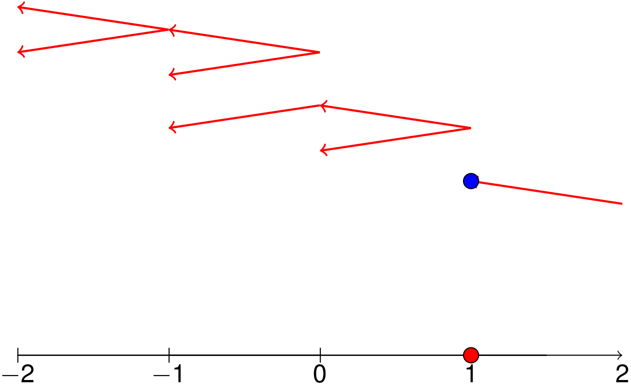

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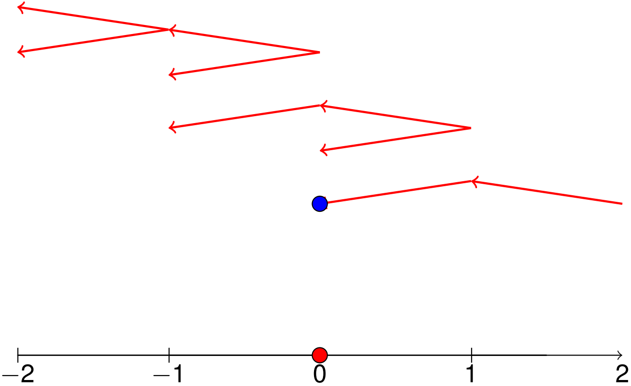

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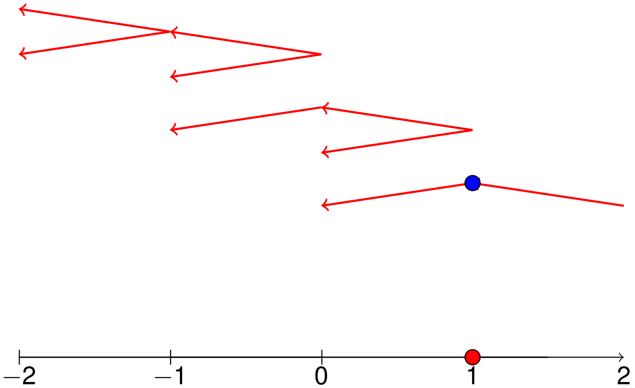
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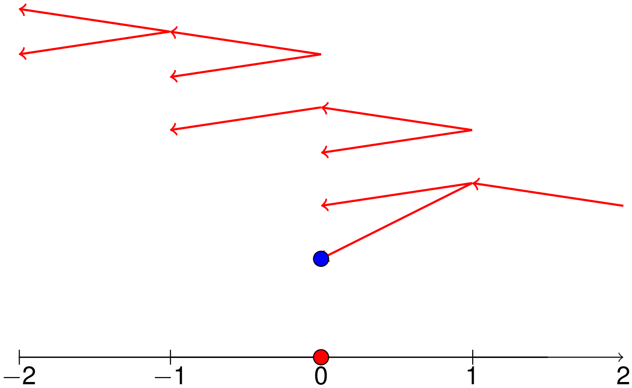
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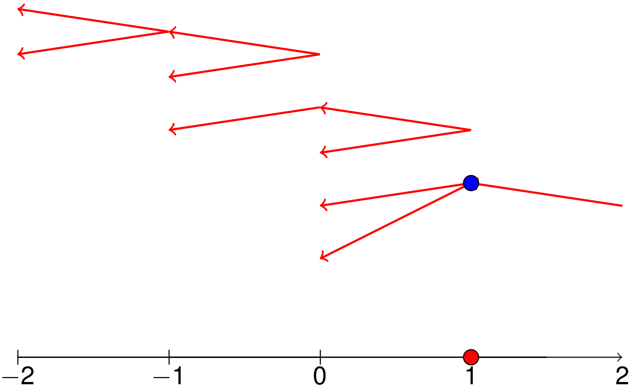
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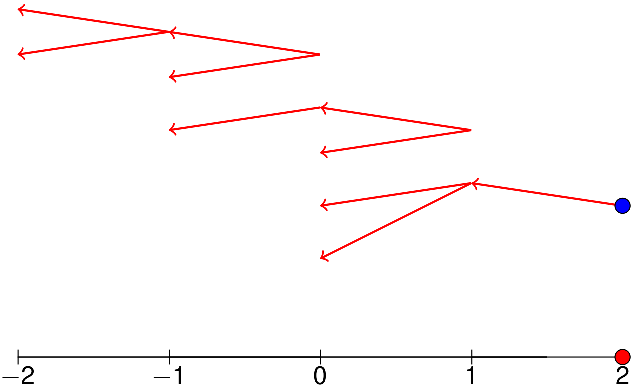
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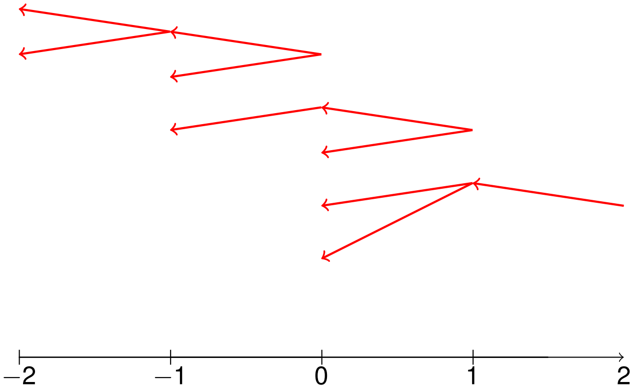
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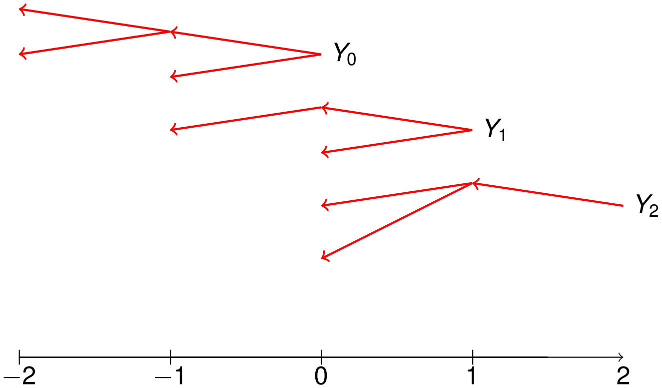

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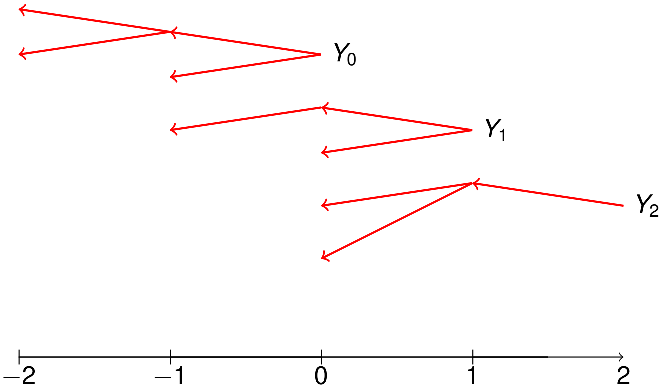

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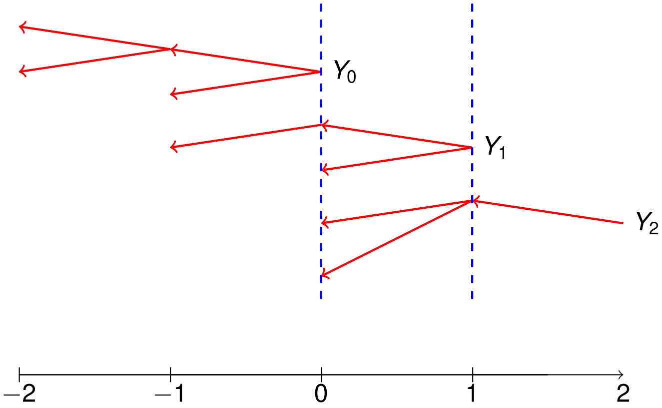

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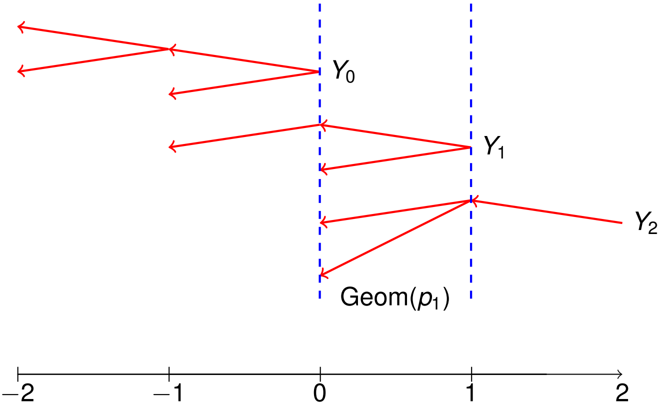

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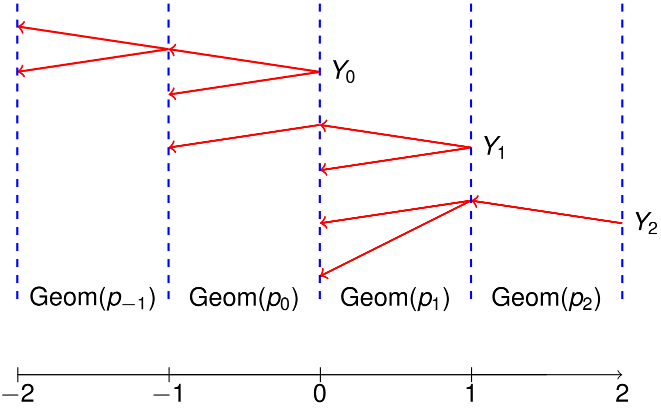

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subcritical BPRE $Y_0 = \sum_{k=1}^{\infty} Z_{0,k}$

$$Z_{0,0} = 1$$

$$Z_{0,k} = \sum_{j=1}^{Z_{0,k-1}+1} V_j^k,$$

where $V_j^k \sim \mathrm{Geom}(p_k)$, $P_\omega[V_j^k = m] = q^m p$, $\mathbb{E}_\omega V_j^k = \frac{q_k}{p_k} = A_k$

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Kesten, Kozlov, Spitzer (Compos. Math.'75)

$$\mathbb{P}[Y_0 = \sum_{k=0}^{\infty} Z_{0,k} > cn] \sim C_Y n^{-\alpha}$$

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Buraczewski, D. (submitted '17) $\rho = \mathbb{E}A^\alpha \log A$,

$$C_Y n^{-\alpha} \sim \mathbb{P}[\sum_{k=0}^{\infty} Z_{0,k} > cn] \sim \mathbb{P}[\sum_{k=n_1}^{n_2} Z_{0,k} > cn],$$

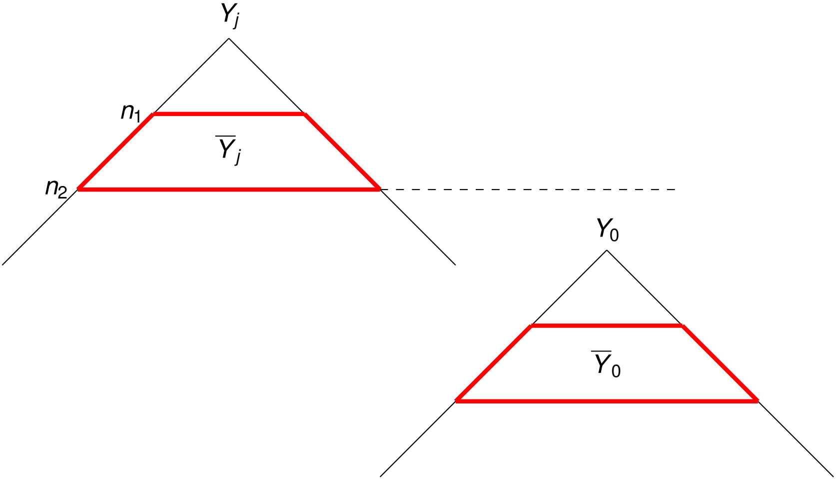
and

$$\mathbb{P}[\sum_{k \notin [n_1,n_2]} Z_{0,k} > cn] = o(n^{-\alpha})$$

where $n_1 = \frac{1}{\rho}\log n - \sqrt{\log n}$, $n_2 = \frac{1}{\rho}\log n + \sqrt{\log n}$.

$$Y_0 \approx \sum_{n_1 \le k < n_2} Z_{0,k} = \overline{Y}_0$$

$$Y_j \approx \sum_{n_1+j \le k < n_2+j} Z_{j,k} = \overline{Y}_j$$



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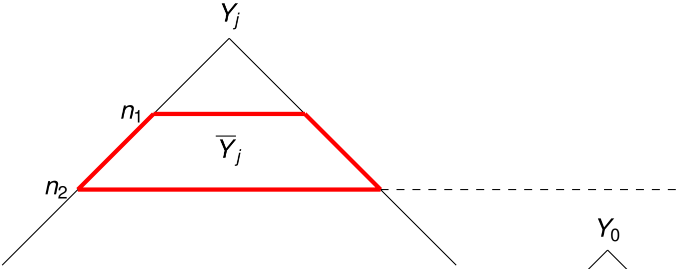

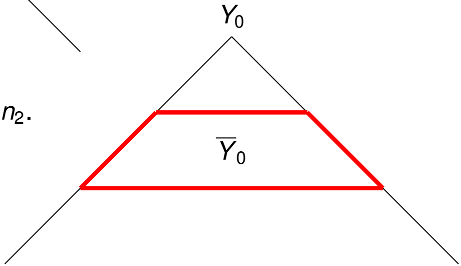

$\overline{Y}_0$ and $\overline{Y}_j$ are independent if $j > n_2$.

$$\mathbb{P}[\sum_{i=0}^{n-1} Y_i > cn]$$

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for $n_1 = \frac{1}{\rho}\log n - \sqrt{\log n}$,

$$\mathbb{P}[\sum_{i=0}^{n-1} Y_i > cn] \sim \mathbb{P}\Big[\sum_{0 \le i \le n} \overline{Y}_i > cn\Big]$$

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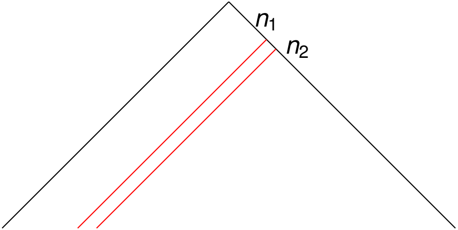

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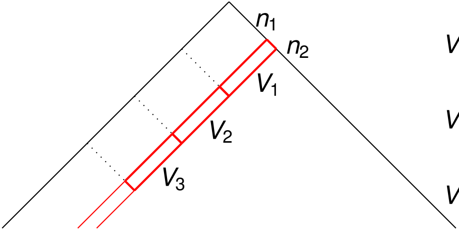

$$V_1 = \sum_{0\le i<n_2} \overline{Y}_i$$

$$V_2 = \sum_{n_2\le i<2n_2} \overline{Y}_i$$

$$V_3 = \sum_{2n_2\le i<3n_2} \overline{Y}_i$$

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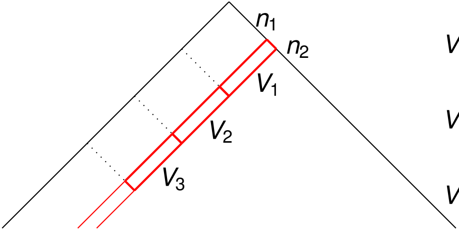

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V_1 depends on $p_0, p_1, \ldots, p_{2n_2-1}$, V_3 depends on $p_{2n_2}, p_{2n_2+1}, \ldots, p_{4n_2-1}$.

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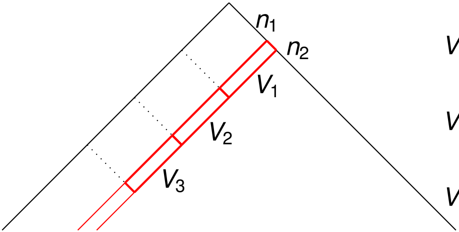

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V_1 and V_3 are independent and identically distributed!

Buraczewski, Damek, Mikosch, Zienkiewicz (Ann. Probab. ‘13)

Precise large deviations for 1-dependent sequence

Buraczewski, Damek, Mikosch, Zienkiewicz (Ann. Probab. '13)

Precise large deviations for 1-dependent sequence

$$\mathbb{P}[\sum_{i=0}^{n-1} Y_i > cn] \sim \mathbb{P}\Big[\sum_{0\le i\le n} \overline{Y}_i > cn\Big] = \mathbb{P}[V_1 + \ldots + V_{\frac{n}{n_2}} > cn]$$

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- V_i are identically distributed and 1-dependent
- $\mathbb{P}[V_i > cn] \sim Cn_2 n^{-\alpha}$ $(V_0 = \sum_{0\le i<n_2} \overline{Y}_i, \mathbb{P}[\overline{Y}_i > n] \sim cn^{-\alpha})$
- $\mathbb{P}[V_i > cn, V_{i+1} > cn] = o\Big(\frac{\sqrt{n_2}}{n^\alpha}\Big)$

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$$\mathbb{P}[\sum_{i=0}^{n-1} Y_i > cn] \sim \mathbb{P}\Big[\sum_{i=1}^{n/n_2} V_i > cn\Big] \sim \frac{n}{n_2}\mathbb{P}[V_1 > cn] \sim Cn^{1-\alpha}$$

Let $A = \frac{q}{p}$. Assume

$$\mathbb{E}A^{\alpha} = 1 \quad \alpha > 1$$

Solomon (Ann. Probab. '75)

$$\lim_{n\to\infty} \frac{X_n}{n} = v = \frac{1 - \mathbb{E}A}{1 + \mathbb{E}A} > 0$$

Kesten, Kozlov, Spitzer (Compos. Math. '75)

$$\alpha \in (1,2), \quad \frac{X_n - vn}{n^{\frac{1}{\alpha}}} \Rightarrow \mathcal{L}_{\alpha}, \qquad \alpha > 2, \quad \frac{X_n - vn}{n^{\frac{1}{2}}} \Rightarrow \mathcal{N}$$

Dembo, Peres, Zeitouni (Commun. Math. Phys. '96)

$$\log \mathbb{P}[X_n < (v - \varepsilon)n] \sim (1 - \alpha) \log n.$$

Buraczewski, D. (submitted '17)

$$\mathbb{P}[X_n < (v - \varepsilon)n] \sim C\varepsilon^{-\alpha} n^{1-\alpha}$$

D. Buraczewski, E. Damek, T. Mikosch, and J. Zienkiewicz.
Large deviations for solutions to stochastic recurrence equations under Kesten's condition. *Ann. Probab, 41(4):2755–2790, 2013*.

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$$C_Y n^{-\alpha} \sim \mathbb{P}[\sum_{k=0}^{\infty} Z_{0,k} > cn] \sim \mathbb{P}[\sum_{k=n_1}^{n_2} Z_{0,k} > cn],$$

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Recall that $\mathbb{E}A^\alpha = 1$ and consider the change of measure

$$\overline{\mathbb{E}}[f(A_k, Z_{0,k})_{k\le n}] = \mathbb{E}\Pi_n^\alpha f(A_k, Z_{0,k})_{k\le n}$$

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Then $\{Z_{0,k}\}$ is supercritical under $\overline{\mathbb{P}}$ since

$$\rho = \overline{\mathbb{E}}\log \mathbb{E}_\omega Z_{0,1} = \overline{\mathbb{E}}\log A_0 = \mathbb{E}A^\alpha \log A > 0$$

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Buraczewski, D. (submitted '17)

$$\frac{\tau_n}{\log(n)}\Big|\, \tau_n < \infty \to^{\mathbb{P}} \frac{1}{\rho}.$$

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