Branching random walks and stretched exponential tails

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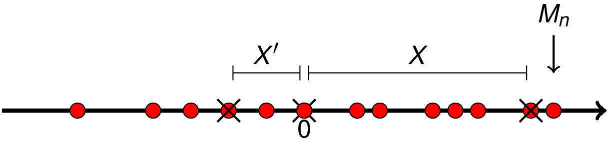
- P. D, N. Gantert and T. Höfelsauer, *The maximum of a branching random walk with stretched exponential tails*, to appear in Ann. inst. Henri Poincare (B) Probab. Stat.

- P. D, N. Gantert and T. Höfelsauer. *Large deviations for the maximum of a branching random walk with stretched exponential tails* Electron. Commun. Probab.

- P. D, N. Gantert, *Extremes of branching random walk with stretched exponential displacements*, arXiv



- the particles reproduce according to a Galton-Watson process $\{Z_n\}_n$ with reproduction mean $\mathbb{E}[Z_1] = m > 1$ such that

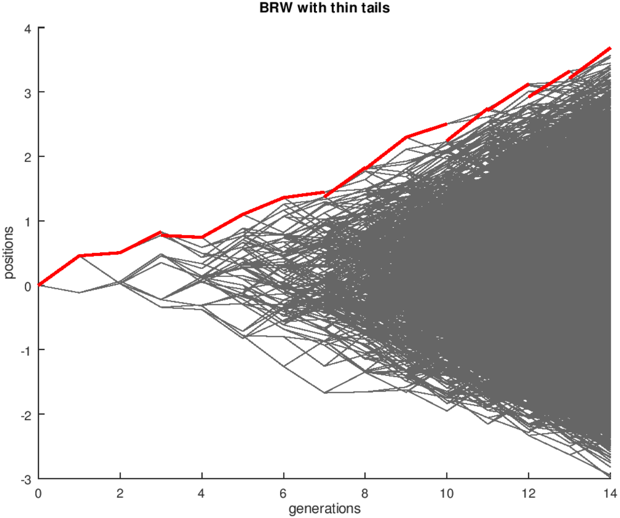
$$\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$$

- the displacements are iid copies of X
- $M_n =$ the position of the rightmost particle
- $\mathbb{P}^*[\cdot] = \mathbb{P}[\cdot \mid \mathrm{NON-EXT}]$

light tails $\mathbb{E}\left[\exp\{\varepsilon|X|\}\right] < \infty$ for some $\varepsilon > 0$

$$M_n - c_1 n - c_2 \log(n) \to^d F_{lt}(x) = \mathbb{E}^*\left[\exp\{-cDe^{-x}\}\right]$$

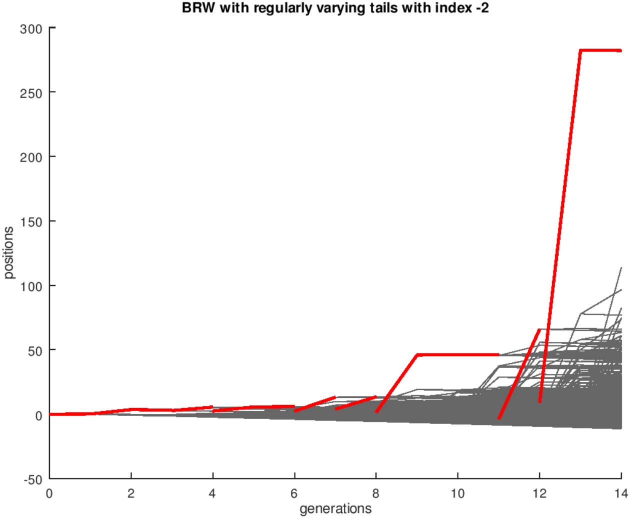
[Biggins '76][Hammersley '74][Kingman '75][Hu & Shi '09][Aïdékon '13]



regularly varying tails $\mathbb{P}[X > t] \sim t^{-\beta}$, $\beta > 0$

$$m^{-n/\beta} M_n \to^d F_{rv}(x) = \mathbb{E}^* \left[\exp\left\{ -cWx^{-\beta} \right\} \right]$$

[Durrett '83]



Theorem ([N. Gantert '00])

Take X such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and

$$\mathbb{P}[X > t] \sim e^{-R(t)}, \quad R(t) = t^r \ell(t), \quad r \in (0,1).$$

$\ell(ct) \sim \ell(t)$. *Put* $d_n = R^{-1}(\log(m)n)$

$$M_n / d_n \to 1$$

Remark

$d_n = n^{1/r} \ell_1(n)$

- if $R(t) = t^r$, then $d_n = (\log(m))^{1/r} n^{1/r}$
- if $R(t) = t^r \log(t)$ then $d_n \sim (r \log(m) n / \log(n))^{1/r}$
- if $R(t) = t^r \log\log(t)$ then $d_n \sim (r \log(m) n / \log\log(n))^{1/r}$

$R(t) = t^r$

$$M_n / n^{1/r} \to \log(m)^{1/r}$$

Theorem ([D, Gantert & Höfelsauer '20])

If $R(t) = t^r$ *then for* $x > \log(m)^{1/r}$,

$$-\frac{1}{n} \log \mathbb{P}[M_n > x n^{1/r}] \to (x^r - \log(m))$$

$$-\frac{1}{n} \log \mathbb{P}[X > x n^{1/r}] = x^r$$

$\{Z_n\}_n$ - Galton-Watson. If $\mathbb{P}[Z_1 \geq 2] < 1$,

$$\frac{1}{n} \log \mathbb{P}^*[Z_n = k] \to -\rho = -\log \mathbb{E}[Z_1 q^{Z_1 - 1}]$$

$q = \mathbb{P}[\mathrm{EXT}]$.

Theorem ([D, Gantert & Höfelsauer '20])

If $R(t) = t^r$ and $X \stackrel{d}{=} -X$, then for $x \in (0, \log(m)^{1/r})$,

$$-\frac{1}{n} \log \mathbb{P}[M_n < xn^{1/r}] \to \begin{cases} k^*(\log(m)^{1/r} - x)^r & \mathbb{P}[Z_1 \geq 2] = 1 \\ \inf_{t \in [0,\dots]} \{t\rho + ((1-t)^{1/r} \log(m)^{1/r} - x)^r\} & \mathbb{P}[Z_1 \geq 2] < 1 \end{cases}$$

$k^* = \min\{k : \mathbb{P}[Z_1 = k] > 0\}$.

Theorem ([D, Gantert '22+])

Take X such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and

$$\mathbb{P}[X > t] \sim e^{-R(t)}, \quad R(t) = t^r \ell(t), \quad r \in (0,1).$$

Put $d_n = R^{-1}(\log(m)n)$. Then there exists $\tau_n \sim nR'(d_n)/2$ such that

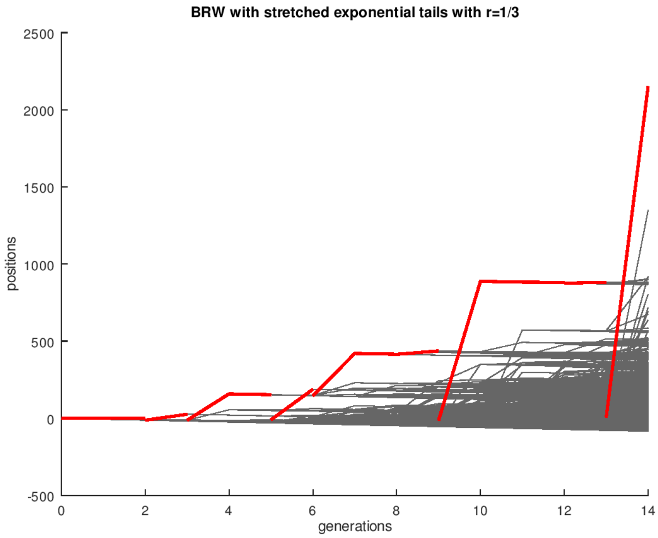
$$\frac{M_n - d_n - \tau_n}{1/R'(d_n)} \to^d F(x) = \mathbb{E}\left[\exp\left\{-\gamma W e^{-x}\right\}\right]$$

where W is a martingale limit associated with the underlying Galton-Watson process and $\gamma > 0$.

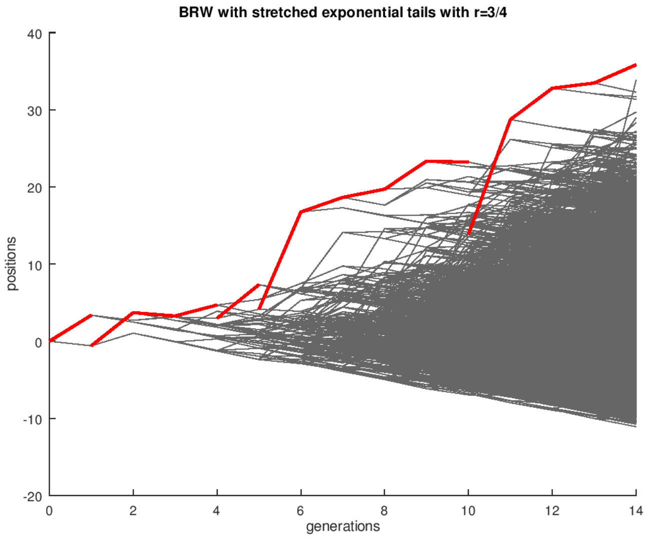
Remark

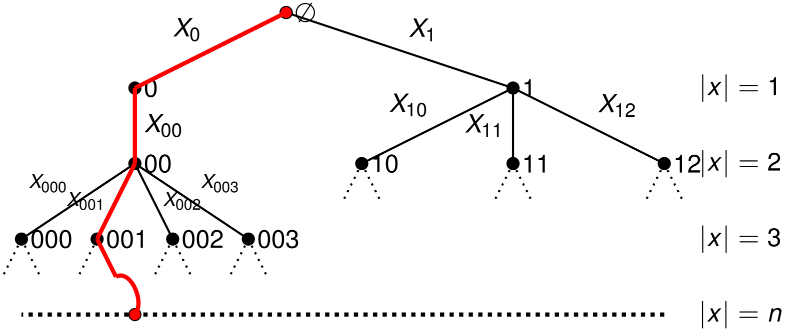
$d_n = n^{1/r}\ell_1(n)$, $1/R'(d_n) = n^{1/r-1}\ell_2(n)$, $\tau_n = n^{2-1/r}\ell_3(n)$

stretched exponential tails $\mathbb{P}[X > t] \sim e^{-R(t)}$, $R(t) = t^r \ell(t)$, $r \leq 2/3$.



stretched exponential tails $\mathbb{P}[X > t] \sim e^{-R(t)}$, $R(t) = t^r \ell(t)$, $r > 2/3$.





X_y, $y \in \mathbb{T}$ iid

For $x \in \mathbb{T}$,

$$V(x) = \sum_{y \le x} X_y$$

$$M_n = \max_{|x|=n} V(x)$$

$$\mathbb{P}[X > t] \sim e^{-R(t)}, \quad R(t) = t^r \ell(t), \quad r \in (0,1).$$

$S_n = X_1 + X_2 + \dots X_n$, $d_n = n^{1/r}\ell_1(n)$.

Theorem ([Nagaev '69], [Denisov, Dieker, Shneer '08])

If $r < 2/3$,

$$\mathbb{P}[S_n > d_n + z] \sim n\mathbb{P}[X > d_n + z]$$

uniformly in $z \in I_n = \dots$.

Remark

$$\mathbb{P}[X > t + x/R'(t) \,|\, X > t] \to e^{-x}$$

If $X > d_n$ then $X - d_n \asymp 1/R'(d_n) = n^{1/r-1}\ell_2(n)$,
If $S_n > d_n$ then $X^* = \max_{k\le n} X_k > d_n$ and so
$X^* - d_n \asymp 1/R'(d_n) = n^{1/r-1}\ell_2(n)$
$S_n \overset{d}{=} X^* + S_{n-1}$
$S_{n-1} \asymp n^{1/2}$

$$\mathbb{P}[X > t] \sim e^{-R(t)}, \quad R(t) = t^r \ell(t), \quad r \in (0,1).$$

$S_n = X_1 + X_2 + \dots X_n$, $d_n = n^{1/r}\ell_1(n)$

$$k_1 = 0, \qquad k_j = \mathbb{E}[X^j] - \sum_{i=1}^{j-1} \binom{j-1}{i} k_{j-i}\mathbb{E}[X^i]$$

$$K(x) = \sum_{j=2}^{\kappa} \frac{k_j}{j!} x^j, \qquad \kappa = \frac{2-r}{1-r}$$

Theorem ([Nagaev ‘69], [D, Gantert ‘22+])

If $r \in (0,1)$, the uniformly in $z \in I_n$,

$$\mathbb{P}[S_n > d_n + z] \sim n e^{-I(d_n+z)}$$

$$I(d_n + z) = \inf_{s \in [0,1]} \{R(d_n + z - nK'(s)) + n(sK'(s) - K(s))\}$$

$$k_1 = 0, \qquad k_j = \mathbb{E}[X^j] - \sum_{i=1}^{j-1} \binom{j-1}{i} k_{j-i} \mathbb{E}[X^i]$$

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Theorem ([Nagaev '69], [D, Gantert '22+])

If $r \geq 2/3$, the uniformly in $z \in I_n$,

$$\mathbb{P}[S_n > d_n + z] \sim n e^{-I(d_n + z)}$$

$$I(d_n + z) = \inf_{s \in [0,1]} \{ R(d_n + z - nK'(s)) + n(sK'(s) - K(s)) \}$$

Remark

If $\mathbb{E}[e^{\varepsilon |X|}] < \infty$ for some $\varepsilon > 0$, then for $\eta(s) = \log \mathbb{E}[e^{sX}]$

$$\mathbb{P}[S_n > \eta'(s) n] \sim \frac{C}{\sqrt{n}} \exp\{ n(\eta(s) - s\eta'(s)) \}$$

$$k_1 = 0, \qquad k_j = \mathbb{E}[X^j] - \sum_{i=1}^{j-1} \binom{j-1}{i} k_{j-i} \mathbb{E}[X^i]$$

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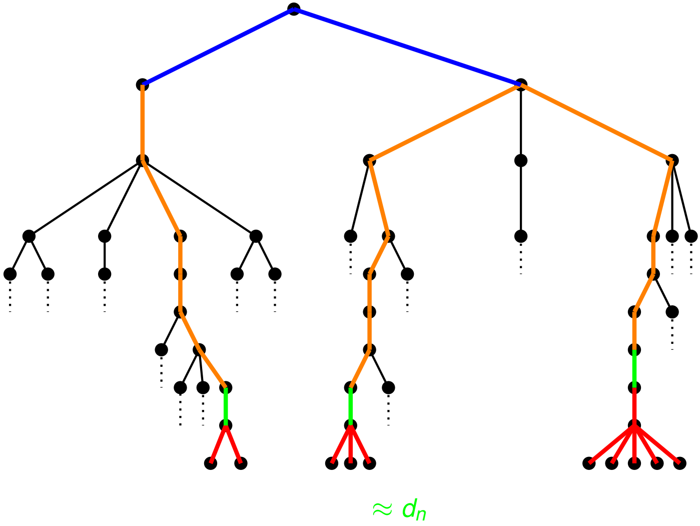
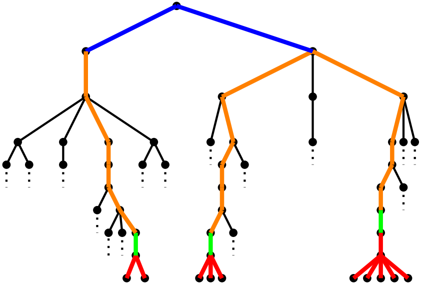
$$\mathbb{P}[S_n > d_n + z] \sim \mathbb{P}[X^* > d_n + z - nK'(s^*)] \mathbb{P}[S_{n-1} > nK'(s^*)]$$

$$s^* \sim R'(d_n + z)$$



$$V(x) = R_1(x) + V_0(x) + N(x) + R_2(x)$$
$$\approx V_0(x) + N(x)$$

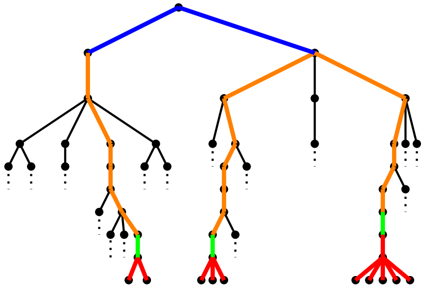


$$M_n \approx \max_{x \in \ldots} \{ V_0(x) + N(x) \}$$

$$\approx d_n + \tau_n + H / R'(d_n)$$

$$R'(d_n)(\max_x N(x) - d_n) \to^d H \stackrel{d}{=} \mathbb{E}[\exp\{-\gamma W e^{-x}\}]$$

$$R'(d_n)(M_n - d_n - \tau_n) \to^d H \stackrel{d}{=} \mathbb{E}[\exp\{-\gamma W e^{-x}\}]$$



$\{y_k\}_k$ - Poisson point process with intensity $e^{-x}\mathrm{d}x$

$$\Lambda = \sum_k T_k \delta_{y_k - \log(\gamma W)} \in \mathcal{M}_p(-\infty, +\infty]$$

$$\mathbb{P}[T_1 = k] = \frac{1}{\gamma} \sum_{i=0}^{\infty} m^{-i} \mathbb{P}[Z_i = k]$$

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Put $d_n = R^{-1}(\log(m)n)$. Then there exists $\tau_n \sim nR'(d_n)/2$ such that

$$\sum_{|x|=n} \delta_{R'(d_n)(V(x) - d_n - \tau_n)} \Rightarrow \Lambda \quad \text{in } \mathcal{M}_p(-\infty, \infty]$$

$\{y_k\}_k$ - Poisson point process with intensity $e^{-x}\mathrm{d}x$

$$\Lambda = \sum_k T_k \delta_{y_k - \log(\gamma W)} \qquad \mathbb{P}[T_1 = k] = \frac{1}{\gamma}\sum_{i=0}^{\infty} m^{-i}\mathbb{P}[Z_i = k]$$

$$W \stackrel{d}{=} \frac{1}{m}\sum_{k=1}^{Z_1} W_k$$

$$\Lambda \stackrel{d}{=} \sum_{k=1}^{Z_1} \Lambda_k - \log(m)$$

Λ from the position of the rightmost particle $\stackrel{d}{=}$

$$\sum_{k=1}^{Z_1} \Lambda_k \text{ from the position of the rightmost particle}$$

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