Tasowanie drzew a hipoteza jakobianowa

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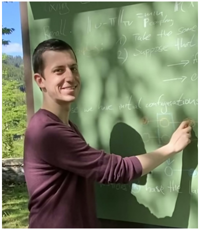

Elia Bisi, P. D., Nina Gantert, Samuel G.G. Johnston, Joscha Prochno, and Dominik Schmid. *Random planar trees and the Jacobian conjecture*, arXiv preprint arXiv:2301.08221 (2025+).

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$$F \circ G = G \circ F = X = (X_1, \ldots, X_n).$$

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Czy jeżeli $F = (F_1, \ldots, F_n)$ jest automorfizmem wielomianowym, to czy każdy F_j jest stopnia jeden?

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Przykład

$$F_1 = X_1 + (X_1 + X_2)^2, \qquad F_2 = X_2 - (X_1 + X_2)^2$$

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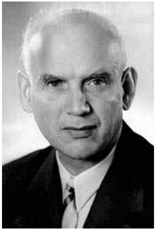

Ganze Cremona-Transformationen.

Von **Ott-Heinrich Keller** in Berlin.

Ph. Furtwängler zum 70. Geburtstage.

Für manche Integritätsbereiche läßt sich eine lineare Basis A $(a_1, a_2, \ldots, a_n)$ angeben, und jedes Element des Integritätsbereiches läßt sich als Linearkombination $\Sigma\lambda_i a_i$ mit ganzzahligen λ_i darstellen. Durch eine unimodulare Lineartransformation erhalten wir eine neue Basis, und umgekehrt hängt jede andere Basis B mit A vermöge einer unimodularen Lineartransformation zusammen. Wir haben also eine vollständige Übersicht über die möglichen Basissysteme.

Nun gibt es Integritätsbereiche, die keine lineare Basis, wohl aber eine Basis $(x_1, x_2, \ldots, x_n)$ besitzen, mit deren Hilfe sich jedes Element als Polynom $\sum \lambda_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$ mit ganzzahligen $\lambda_{i_1 \ldots i_n}$ darstellen läßt. Ein Beispiel sind die symmetrischen Funktionen; für sie sind die symmetrischen Grundfunktionen eine solche Basis. Es ergibt sich die Aufgabe, auch hier eine Übersicht über alle möglichen Basissysteme und die vermittelnden Transformationen zu gewinnen.

Ott-Heinrich Keller *Ganze Cremona-Transformationen*. Monatsch. Math. Phys. 47: 299–306, 1939

Es sind nur die Fälle 6., 7. noch unentschieden, also die Frage, ob Polynome mit der Funktionaldeterminante 1 sich stets durch Polynome umkehren lassen. Hier sind die Voraussetzungen im gegebenen Fall am einfachsten nachzuprüfen, da man dabei über die Umkehrung nichts zu wissen braucht. Mir scheint die Frage eine Untersuchung sehr zu lohnen, sie scheint jedoch bereits im ebenen Fall sehr schwierig zu sein.

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$n \geq 3$: problem otwarty

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Twierdzenie (Bass, Connell, Wright 1982)

Dla rozstrzygnięcia hipotezy o jakobianie wystarczy rozważać F postaci

$$F_j = X_j - H_j, \qquad j = 1, \ldots, n,$$

gdzie $H_j \in \mathbb{C}[X_1, \ldots, X_n]$ są d-jednorodne, $d \in \mathbb{N}$.

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JH jest nilpotentna

Problem

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$$F_j = X_j - H_j, \qquad H_j \text{ jednorodne stopnia } d, \qquad JH \text{ nilpotentna}$$

Czy $F^{-1} \in \mathbb{C}[X_1, \ldots, X_n]^n$?

Definicja

Dla $p \in \mathbb{N}$ niech C_p oznacza zbiór wszystkich ukorzenionych, planarnych, pełnych drzew d-arnych o p liściach (każdy wewnętrzny wierzchołek ma dokładnie d dzieci).

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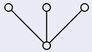
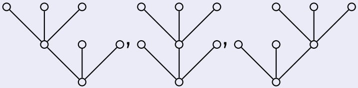
$C_3 = \Big\{$  $\Big\}$

$C_5 = \Big\{$  $\Big\}$

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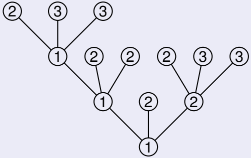

$\in \mathcal{C}_9^{(1)}$

Definicja

$$\omega(c,T) = \prod_{v\in V_{\mathrm{int}}(T)} \frac{h_{\kappa(v)}^{(c(v))}}{d!} \prod_{v\in V_{\mathrm{ext}}(T)} X_{c(v)} \in \mathbb{C}[X_1,\ldots,X_n]$$

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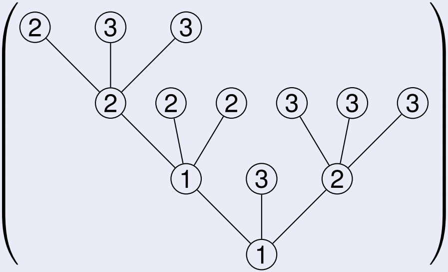
ω



$$= \frac{h^{(1)}_{111}}{3!} \frac{h^{(1)}_{030}}{3!} \frac{h^{(2)}_{003}}{3!} \frac{h^{(2)}_{012}}{3!} X_2^3 X_3^6$$

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$$\mathcal{C}^{(j)} = \{ \; j \; \} \cup \bigcup_{\beta \in [n]^d} \left\{ \begin{array}{ccc} \mathcal{T}_1 & \cdots & \mathcal{T}_d \\ & j & \end{array} \ : \ \mathcal{T}_k \in \mathcal{C}^{(\beta_k)} \right\}$$

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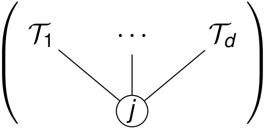
$$G_j = \sum_{\mathcal{T} \in \mathcal{C}^{(j)}} \omega(\mathcal{T}) = \omega\left(\textcircled{j}\right) + \sum_{\substack{\beta \in [n]^d \\ \mathcal{T}_1 \in \mathcal{C}^{(\beta_1)}, \ldots, \mathcal{T}_d \in \mathcal{C}^{(\beta_d)}}} \omega$$



$$= X_j + \sum_{\substack{\beta \in [n]^d \\ \mathcal{T}_1 \in \mathcal{C}^{(\beta_1)}, \ldots, \mathcal{T}_d \in \mathcal{C}^{(\alpha_d)}}} \frac{h^{(j)}_{\kappa(root)}}{d!} \omega(\mathcal{T}_1) \cdots \omega(\mathcal{T}_d)$$

$$H_j = \sum_{|\alpha|=d} \frac{h_\alpha^{(j)}}{\alpha!} X^\alpha, \qquad F_j = X_j - H_j$$

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$$= X_j + \sum_{\substack{\beta \in [n]^d \\ \mathcal{T}_1 \in \mathcal{C}^{(\beta_1)}, \dots, \mathcal{T}_d \in \mathcal{C}^{(\alpha_d)}}} \frac{h_{\kappa(root)}^{(j)}}{d!} \omega(\mathcal{T}_1) \cdots \omega(\mathcal{T}_d)$$

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$$= X_j + \sum_{|\alpha|=d} \frac{h_\alpha^{(j)}}{\alpha!} G_1^{\alpha_1} \cdots G_d^{\alpha_d} = X_j + H_j \circ G$$

$$\omega_j(T) = \sum_{c\,:\,(c,T)\in\mathcal{C}^{(j)}} \omega(c, T)$$

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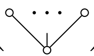
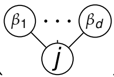
$\omega_j \Bigg($  $\Bigg) = \displaystyle\sum_{\beta\in[n]^d} \omega \Bigg($  $\Bigg) = \displaystyle\sum_{\beta\in[n]^d} \frac{h^{(j)}_{\kappa(root)}}{d!} X_{\beta_1} X_{\beta_2} \cdots X_{\beta_n}$

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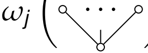
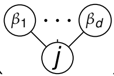
$\omega_j \Big($  $\Big) = \sum_{\beta\in[n]^d} \omega \Big($ 

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$$G_j = \sum_{p} \sum_{T\in C_p} \omega_j(T) \in \mathbb{C}[X_1,\ldots,X_n]$$

wtedy i tylko wtedy, gdy

$$\sum_{T\in C_p} \omega_j(T) = 0 \qquad \text{dla dostatecznie dużych } p$$

$$\omega(c, T) = \prod_{v \in V_{\mathrm{int}}(T)} \frac{h_{\kappa(v)}^{(c(v))}}{d!} \prod_{v \in V_{\mathrm{ext}}(T)} X_{c(v)},$$

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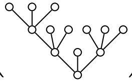
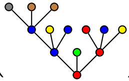
$\omega_j \Bigg($  $\Bigg) = \sum_{c} \omega \Bigg($  $\Bigg)$

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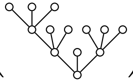
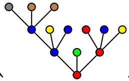
$\omega_j \Bigg($  $\Bigg) = \sum_c \omega \Bigg($  $\Bigg)$

Definicja

Rozważmy $T \in C_p$ z wyróżnionym liściem $v \in V_{\mathrm{ext}}(T)$.

$$\omega_{ji}(T,v) = \sum_{\substack{c\,:\,(T,c)\in\mathcal{C}^{(j)} \\ c(root)=j,\ c(v)=i}} \prod_{u\in V_{\mathrm{int}}(T)} \frac{h^{(c(u))}_{\kappa(u)}}{d!} \prod_{u\in V_{\mathrm{ext}}(T)\setminus\{v\}} X_{c(u)}$$

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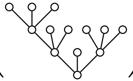
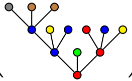
$\omega_j \Bigg($  $\Bigg) = \sum_c \omega \Bigg($  $\Bigg)$

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Lemat

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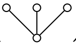
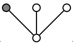
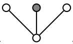
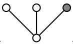
$\frac{\partial H_j}{\partial X_i} = \frac{\partial}{\partial X_i}\omega_j \Big($  $\Big) = \omega_{ji} \Big($  $\Big) + \omega_{ji} \Big($  $\Big) + \omega_{ji} \Big($  $\Big)$

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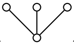
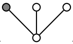
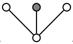
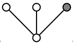
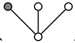
$\frac{\partial H_j}{\partial X_i} = \frac{\partial}{\partial X_i}\omega_j \Bigg($  $\Bigg) = \omega_{ji} \Bigg($  $\Bigg) + \omega_{ji} \Bigg($  $\Bigg) + \omega_{ji} \Bigg($  $\Bigg)$

$= 3\omega_{ji} \Bigg($  $\Bigg)$

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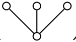
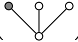
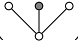
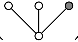
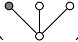
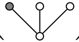
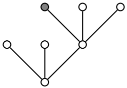
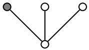
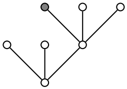
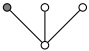
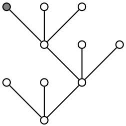
$\frac{\partial H_j}{\partial X_i} = \frac{\partial}{\partial X_i}\omega_j\Big($  $\Big) = \omega_{ji}\Big($  $\Big) + \omega_{ji}\Big($  $\Big) + \omega_{ji}\Big($  $\Big)$

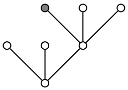
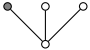
$= 3\omega_{ji}\Big($  $\Big)$

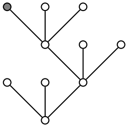
$JH = 3\omega\Big($  $\Big)$

$(S, u) =$  $\quad (T, v) =$ 

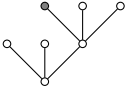
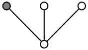
$(S, u) =$  $(T, v) =$ 

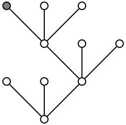
$(S, u)(T, v) =$ 

$(S, u) =$  $\quad (T, v) =$ 

$(S, u)(T, v) =$ 

$$\omega_{ji}\big((S, u)(T, v)\big) = \sum_{k=1}^{n} \omega_{jk}(S, u)\, \omega_{ki}(T, v)$$

$(S, u) =$  $\quad (T, v) =$ 

$(S, u)(T, v) =$ 

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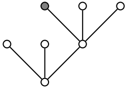
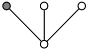
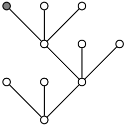
$(S, u) =$  $\qquad (T, v) =$ 

$(S, u)(T, v) =$ 

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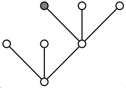
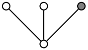
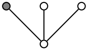
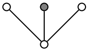
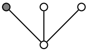
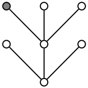
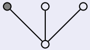
$$\omega\big((S, u)(T, v)\big) = \omega(S, u)\,\omega(T, v)$$

$JH^2 = 9\omega \Bigg($  $\Bigg) = 9\omega \Big($  $\Big)\,\omega \Big($  $\Big)$

$= 9\omega \Big($  $\Big)\,\omega \Big($  $\Big) = 9\omega \Bigg($  $\Bigg)$

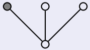
Definicja

Paprocią rzędu k nazywamy produkt k grafów postaci



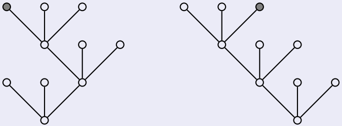
Definicja

Paprocią rzędu *k* nazywamy produkt *k* grafów postaci



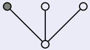
Przykład

Paprocie rzędu 3:



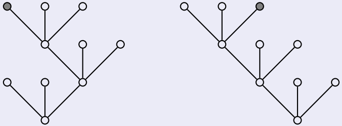
Definicja

Paprocią rzędu k nazywamy produkt k grafów postaci



Przykład

Paprocie rzędu 3:



Jeżeli U jest paprocią rzędu n, to $\omega_{ij}(U) = 0$ dla każdych $i, j \leq n$.

Cel

$$\sum_{T \in C_p} \omega_j(T) = 0$$

dla dostatecznie dużych p.

$T \in C_p$ i $v \in V(T)$ w odległości $\geq n$ od korzenia. Rozważmy ścieżkę przodków $v = v_n, v_{n-1}, \ldots v_0$

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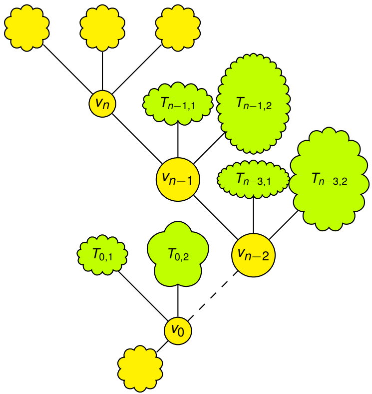
$T =$ 

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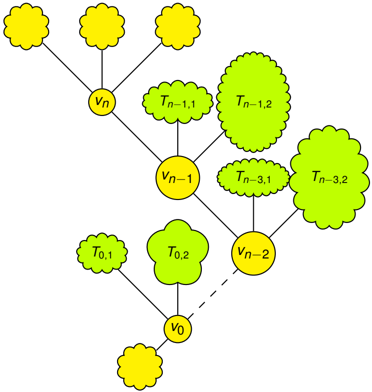
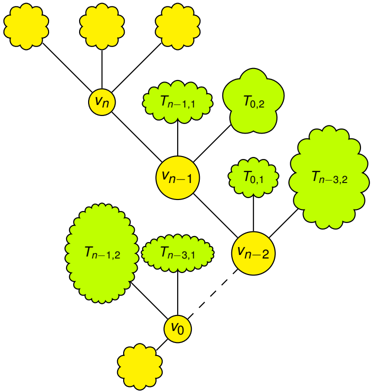


$T' =$



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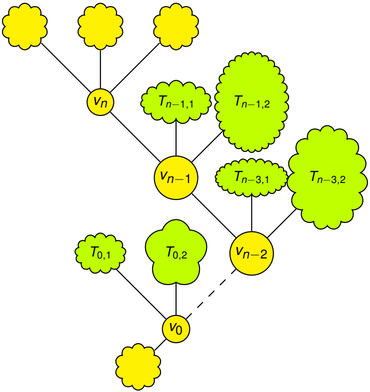
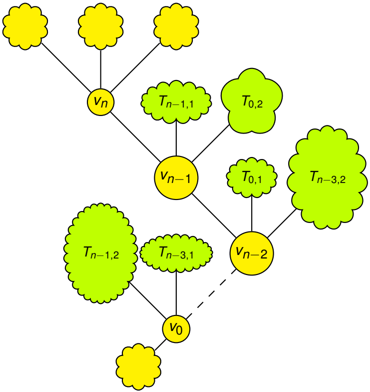
$T =$ 

$T' =$ 

Definicja

Shuffle class $Sh(T, v)$ składa się ze wszystkich drzew T', które można uzyskać w ten sposób

Lemat

Dla każdego T i każdego $v \in V(T)$ w odległości $\geq n$ od korzenia

$$\sum_{T' \in Sh(T,v)} \omega_j(T') = 0$$

Lemat

Dla każdego T i każdego $v \in V(T)$ w odległości $\geq n$ od korzenia

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Hipoteza (o tasowaniu poddrzew)

Dla dostatecznie dużych p funkcja stała na C_p leży w

$$\mathrm{span}\left\{ \mathbf{1}_{\mathcal{S}} \ : \ \mathcal{S} \text{ jest shuffle class w } C_p \right\},$$

$$\mathbf{1}_{\mathcal{S}}(T) = \begin{cases} 1 & T \in \mathcal{S} \\ 0 & T \notin \mathcal{S} \end{cases}$$

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Twierdzenie

Hipoteza o tasowaniu poddrzew implikuje hipotezę o jakobianie.

$$\sum_{T' \in Sh(T,v)} \omega_j(T') = 0$$

$$1 = \sum_{\mathcal{S}} \lambda_{\mathcal{S}} \mathbf{1}_{\mathcal{S}}(T), \qquad T \in C_p$$

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$$0 = \sum_{\mathcal{S}} \lambda_{\mathcal{S}} \langle \mathbf{1}_{\mathcal{S}}, \omega_j \rangle = \langle 1, \omega_j \rangle = \sum_{T \in C_p} \omega_j(T)$$

Hipoteza (o tasowaniu poddrzew)

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Twierdzenie ($EBP^2D^2NGS^2J^2$)

Istnieją funkcje

$$\psi, \phi \in \text{span}\left\{ \mathbf{1}_{\mathcal{S}} \ : \ \mathcal{S} \text{ jest shuffle class w } C_p \right\},$$

takie, że

$$\|1 - \psi\|_\infty \le e^{-C\log(p)^2}, \qquad \|1 - \phi\|_1 \le e^{-Cp} \# C_p$$

dla dużych p

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Definicja

Drzewo T jest n-doskonałe jeżeli zawiera scieżkę $v_0, \dots v_n$ taką, że każde z $(d-1)n$ rodzeństwa $v_0, \dots, v_{n-1}$ to liście.

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Jeżeli T jest n-doskonałe, to $Sh(T, v_n) = \{T\}$.

$$
\begin{aligned}
c_p = \#C_p = \frac{1}{p}\binom{dk}{k}, \qquad p = (d-1)k+1 \\
= Cp^{-3/2}e^{Cp}(1+o(1))
\end{aligned}
$$

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$$\mathbb{P}[(P_1,\ldots,P_d) = (p_1,\ldots,p_d)] = \frac{c_{p_1}c_{p_2}\cdots c_{p_d}}{c_p}$$

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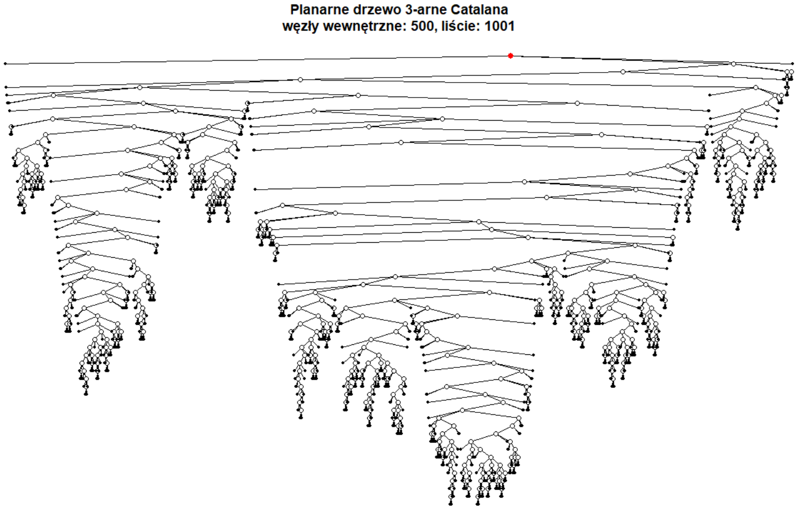
$(Y_1, \ldots, Y_{d-1})$ powstaje z $(P_1, \ldots, P_d)$ poprzez usunięcie największej wartości

$$
\begin{aligned}
\mathbb{P}[(Y_1, \ldots, Y_{d-1}) = (y_1, \ldots, y_{d-1})] &= d c_{y_1} c_{y_2} \cdots c_{y_{d-1}} \frac{c_{p - y_1 - \ldots - y_{d-1}}}{c_p} \\
&\to d c_{y_1} c_{y_2} \cdots c_{y_{d-1}} e^{-C(y_1 + \ldots + y_{d-1})}
\end{aligned}
$$


Planarne drzewo 3-arne Catalana
węzły wewnętrzne: 500, liście: 1001

Niech $\mathbb{T}_p$ będzie losowym drzewem z C_p.

Lemat

$$\mathbb{P}\left[\text{w } \mathbb{T}_p \ d-1 \text{ dzieci korzenia to liście}\right] \geq 1/e$$

Twierdzenie

$$\mathbb{P}[\mathbb{T}_p \text{ nie jest } n\text{-doskonałe}] \leq e^{-Cp}$$

$$\phi(T) = \mathbf{1}_{T \text{ jest } p\text{-doskonałe}} \in \operatorname{span}\left\{ \mathbf{1}_{\mathcal{S}} \ : \ \mathcal{S} \text{ jest shuffle class w } C_p \right\},$$

$$(\# C_p)^{-1} \| 1 - \phi \|_1 = \mathbb{P}[\mathbb{T}_p \text{ nie jest } n\text{-doskonałe}] \leq e^{-Cp}$$

$$H_j = \sum_{|\alpha|=d} \frac{h_\alpha^{(j)}}{\alpha!} X^\alpha, \qquad G_j = \sum_\alpha \frac{g_\alpha^{(j)}}{\alpha!} X^\alpha$$

Twierdzenie (*$EBP^2D^2NGS^2J^2$*)

Niech $F = (F_1, \ldots, F_n) \in \mathbb{C}[X_1, \ldots, X_n]$ będzie postaci

$$F_j = X_j - H_j, \qquad H_j \ d\text{-jednorodne}$$

Niech $L = \sup_{\alpha,j} h_\alpha^{(j)}$. $|\alpha| = (d-1)k + 1$

$$|g_\alpha^{(j)}| \leq \frac{1}{(d!)^k} \frac{1}{(d-1)k+1} \binom{dk}{k} n^{k-1} \frac{|\alpha|!}{\alpha!} L^k$$

Jeżeli $\det(JF) = 1$, to JH jest nilpotentna i wtedy

$$|g_\alpha^{(j)}| \leq \frac{1}{(d!)^k} \frac{1}{(d-1)k+1} \binom{dk}{k} n^{k-1} \frac{|\alpha|!}{\alpha!} L^k \times Ce^{-ck}.$$