Kinetic-type equations with perturbed collisions

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Buraczewski, Dariusz, P. D, and Alexander Marynych. "Solutions of kinetic-type equations with perturbed collisions." Stochastic Processes and their Applications 159 (2023): 199-224.

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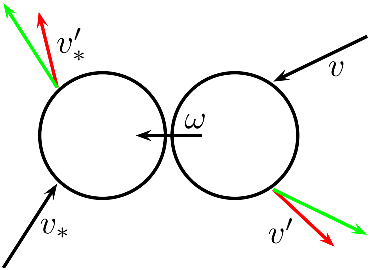
$$\partial_t f + v \cdot \nabla_x f = 0$$

$v, v_* \in \mathbb{R}^3$



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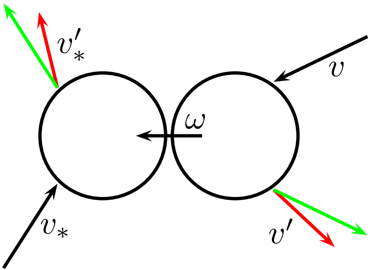

$$v' + v'_* = v + v_*$$

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-e^2}{2}|(v - v_*) \cdot \omega|^2 \leq 0$$

$e \in (0, 1]$ is the restitution coefficient

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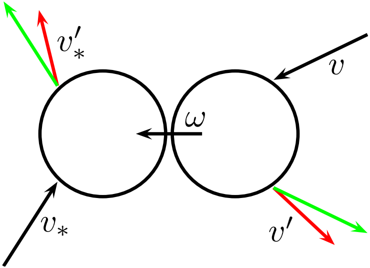

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$$v' = \frac{1}{2}(v + v_*) - \frac{1-e}{4e}(v - v_*) + \frac{1+e}{4e}|v - v_*|\omega$$

$$v'_* = \frac{1}{2}(v + v_*) + \frac{1-e}{4e}(v - v_*) + \frac{1+e}{4e}|v - v_*|\omega$$

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$f = f(t, v)$, $v \in \mathbb{R}$

$$Q(f, f) = Q_+(f, f) - f$$

$$\int \psi(v) Q_+(f,f)(v) \mathrm{d}v = \mathbb{E}\left[\int \int \psi(A_1 v + A_2 v_*) f(v) f(v_*) \mathrm{d}v \mathrm{d}v_* \right]$$

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- F. Bassetti and L. Ladelli, *Self-similar solutions in one-dimensional kinetic models: A probabilistic view*, Ann. Appl. Probab. **22** (2012), no. 5, 1928–1961.
- K. Bogus, D. Buraczewski, and A. Marynych, *Self-similar solutions of kinetic-type equations: the boundary case*, Stoch. Proc. Appl. **130** (2020), no. 2, 677–693.
- D. Buraczewski, K. Kolesko, and M. Meiners, *Self-similar solutions to kinetic-type evolution equations: beyond the boundary case*, Electron. J. Probab. **26** (2021), 1 – 18.

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F. Bassetti, L. Ladelli, and G. Toscani, *Kinetic models with randomly perturbed binary collisions*, J. Stat. Physics **142** (2011), no. 4, 686–709.

D. Buraczewski, P. D, and A. Marynych. *Solutions of kinetic-type equations with perturbed collisions.* Stochastic Processes and their Applications **159** (2023): 199-224.

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$$\varphi_t(\xi) = \int_{\mathbb{R}} e^{iv\xi} f(t, v) \mathrm{d}v$$

$$\partial_t \varphi_t + \varphi_t = \widehat{Q_+}(\varphi_t, \varphi_t)$$

$$\widehat{Q_+}(\varphi_t, \varphi_t)(\xi) = \mathbb{E}\left[\varphi_t(A_1 \xi) \varphi_t(A_2 \xi) e^{i\xi C}\right]$$

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If $\varphi_t(\xi) \to \varphi_\infty(\xi)$, then

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$\varphi_\infty(\xi) = \mathbb{E}[e^{i\xi W}]$

$$W \stackrel{d}{=} A_1 W_1 + A_2 W_2 + C$$

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$$w_\infty(\xi) = \int_0^1 \widehat{Q}_0(w_\infty)(s^{-\alpha}\xi)\mathrm{d}s = \mathbb{E}\left[w_\infty(U^{-\alpha}A_1\xi)w_\infty(U^{-\alpha}A_2\xi)\right]$$

$w_\infty(\xi) = \mathbb{E}[e^{i\xi W}]$

$$W \overset{d}{=} U^{-\alpha}A_1W_1 + U^{-\alpha}A_2W_2$$

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$$\varphi_t = e^{-t}\varphi_0 + \int_0^t e^{-s}\widehat{Q_+}(\varphi_{t-s}, \varphi_{t-s})\mathrm{d}s$$

$$\varphi_t(\xi) = \mathbb{E}\left[\mathbb{1}_{\{E>t\}}\varphi_0(\xi) + \mathbb{1}_{\{E\leq t\}}e^{i\xi C}\varphi_{t-E}(A_1\xi), \varphi_{t-E}(A_2\xi)\right]$$

$\mathbb{T}_2$ - infinite binary tree. $\{(A_1(v), A_2(v), C(v)\}_{v\in\mathbb{T}_2}$ iid

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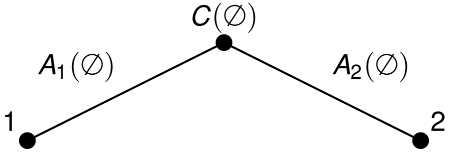

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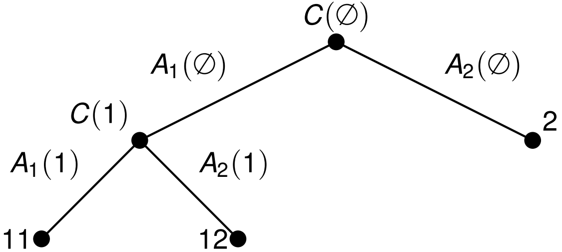

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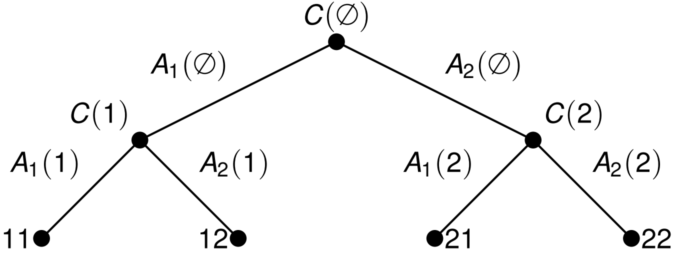

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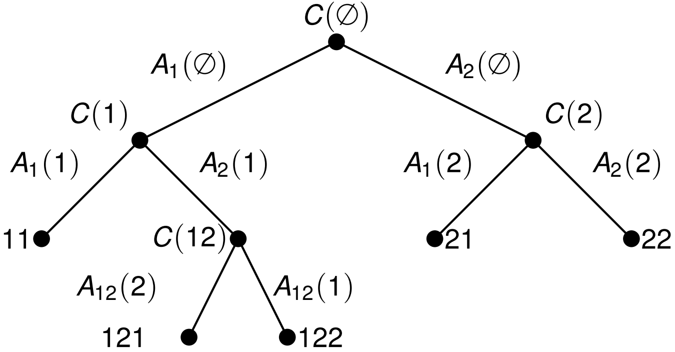

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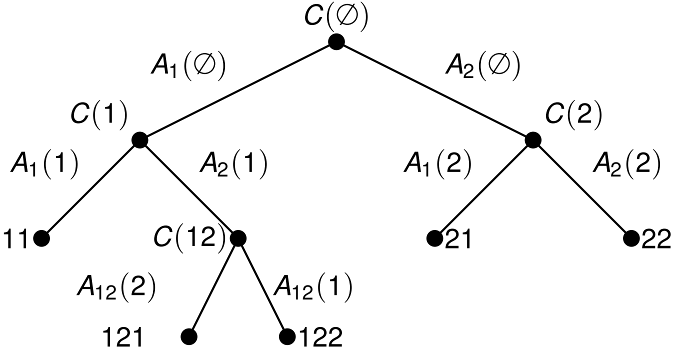

$\partial\mathcal{T}_t = \{\text{nodes active at time } t\}$ $\quad$ $\mathcal{T}_t = \{\text{nodes inactivated by time } t\}$

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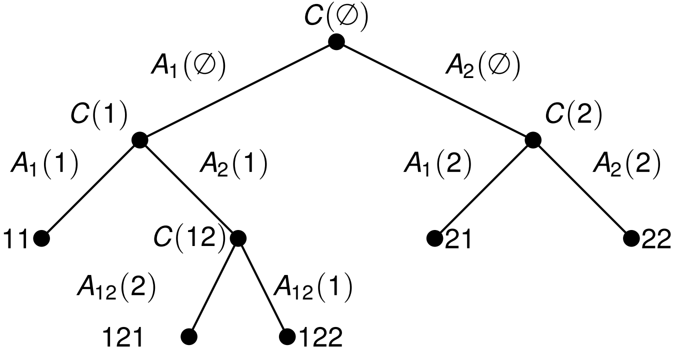

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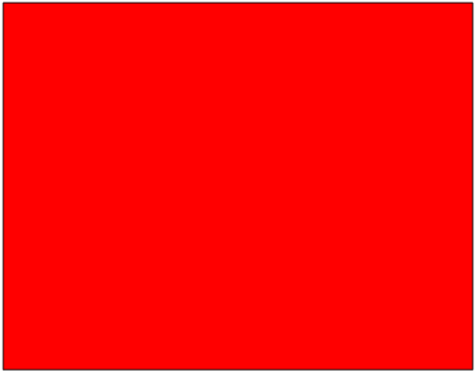
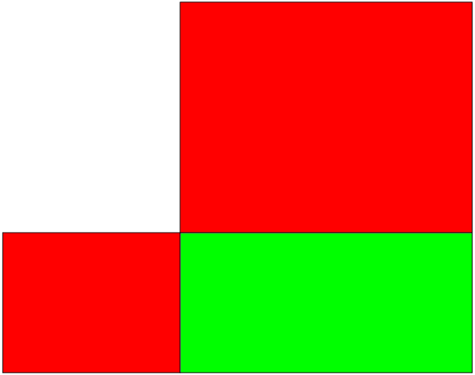
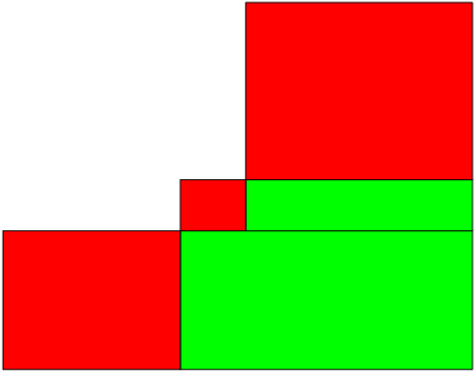
on $\{\emptyset \in \mathcal{T}_t\}$

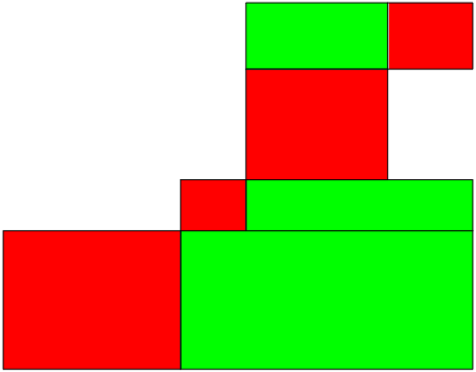
$$\sum_{v \in \partial \mathcal{T}_t} L(v) X(v) = A_1(\emptyset) \sum_{1 \le v \in \partial \mathcal{T}_t} \frac{L(v)}{A_1(\emptyset)} X(v) + A_2(\emptyset) \sum_{2 \le v \in \partial \mathcal{T}_t} \frac{L(v)}{A_2(\emptyset)} X(v)$$

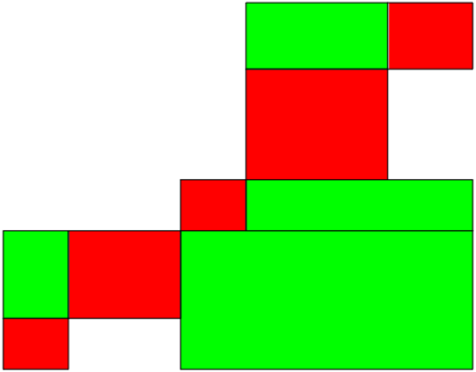
$$\sum_{v \in \mathcal{T}_t} L(v) C(v) = C_1(\emptyset) + A_1(\emptyset) \sum_{1 \le v \in \mathcal{T}_t} \frac{L(v)}{A_1(\emptyset)} C(v) + A_2(\emptyset) \sum_{2 \le v \in \mathcal{T}_t} \frac{L(v)}{A_2(\emptyset)} C$$

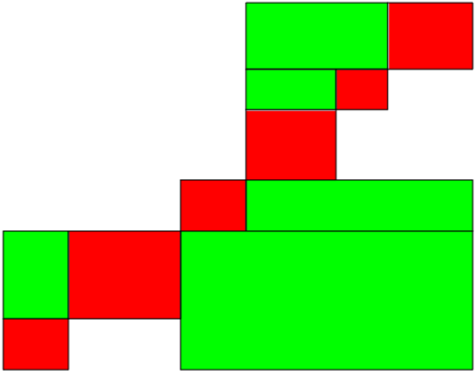


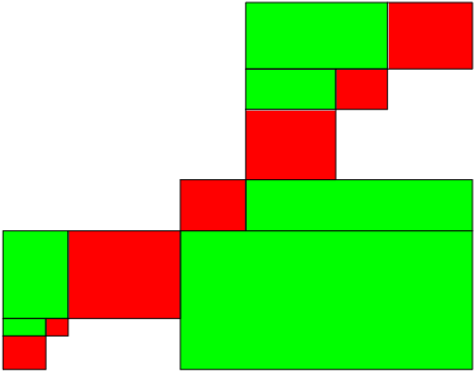


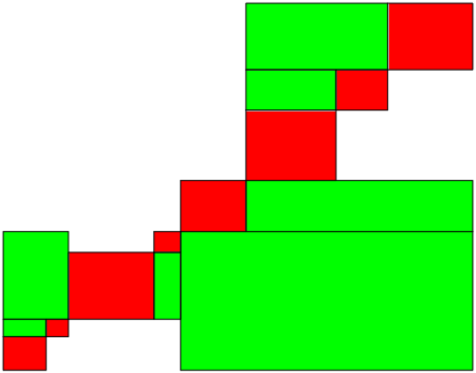


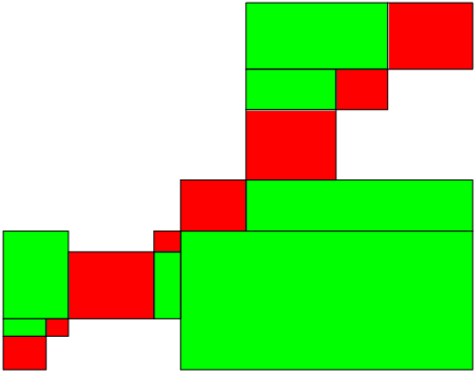












$$W_t = \sum_{v \in \partial \mathcal{T}_t} L(v) X(v) + \sum_{v \in \mathcal{T}_t} L(v) C(v) = \text{red area} + \text{green area}$$

$$A_1 = U^2, \quad A_2 = (1-U)^2, \quad C = U(1-U), \qquad X = 1.$$

$$W_t = \sum_{v \in \partial \mathcal{T}_t} L(v) X(v) + \sum_{v \in \mathcal{T}_t} L(v) C(v)$$

$$\varphi_t(\xi) = \mathbb{E}\left[e^{i\xi W_t} \right]$$

$$\varphi_{t+h}(\xi) = \mathbb{E}[\mathbb{1}_{\{\text{ no splits during [0, h] }\}} \varphi_t(\xi)] + \mathbb{E}\left[\mathbb{1}_{\{\text{one split during } [0, h]\}} \varphi_t(A_1 \xi) \varphi_t(A_2 \xi) e^{i\xi C} \right] + o(h)$$

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$(\Pi_n)_{n\in\mathbb{N}}$ - multiplicative random walk

$$\lambda(\alpha) = \log \mathbb{E}\left[A_1^\alpha + A_2^\alpha\right]$$

$$\mathbb{E}[h(\Pi_1)] = e^{-\lambda(\alpha)}\mathbb{E}\left[A_1^\alpha h(A_1) + A_2^\alpha h(A_2)\right]$$

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$$\mathbb{E}\left[\sum_{v\in\partial\mathcal{T}_t} f(L(v))\right] = \mathbb{E}\left[\Pi_{N_t}^{-\alpha} e^{\lambda(\alpha)N_t} f(\Pi_{N_t})\right]$$

$(N_t)_t$ - homogeneous Poisson process

$$\mathbb{E}\left[\sum_{v\in\partial\mathcal{T}_t} L(v)^\gamma\right] = e^{t\Phi(\gamma)}, \quad \Phi(\gamma) = \mathbb{E}[A_1^\gamma + A_2^\gamma] - 1$$

$$M_t(\gamma) = e^{-t\Phi(\gamma)} \sum_{v \in \partial \mathcal{T}_t} L(v)^\gamma$$

$s < t$

$$\mathbb{E}[M_t | \mathcal{F}_s] = e^{-t\Phi(\gamma)} \sum_{u \in \partial \mathcal{T}_s} L(u)^\gamma \mathbb{E}\left[\sum_{u \geq v \in \partial \mathcal{T}_t} (L(v)/L(u))^\gamma \middle| \mathcal{F}_s \right]$$

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$$= e^{-t\Phi(\gamma)} \sum_{u \in \partial \mathcal{T}_s} L(u)^\gamma e^{(t-s)\Phi(\gamma)} = M_s$$

$$M_t(\gamma) = e^{-t\Phi(\gamma)} \sum_{v \in \partial \mathcal{T}_t} L(v)^\gamma$$

$s < t$

$$\begin{aligned}
\mathbb{E}[M_t | \mathcal{F}_s] &= e^{-t\Phi(\gamma)} \sum_{u \in \partial \mathcal{T}_s} L(u)^\gamma \mathbb{E}\left[\sum_{u \geq v \in \partial \mathcal{T}_t} (L(v)/L(u))^\gamma \middle| \mathcal{F}_s \right] \\
&= e^{-t\Phi(\gamma)} \sum_{u \in \partial \mathcal{T}_s} L(u)^\gamma \mathbb{E}\left[\sum_{u \geq v \in \partial \mathcal{T}_t} (L(v)/L(u))^\gamma \right] \\
&= e^{-t\Phi(\gamma)} \sum_{u \in \partial \mathcal{T}_s} L(u)^\gamma e^{(t-s)\Phi(\gamma)} = M_s
\end{aligned}$$

$$M_t(\gamma) \to M_\infty(\gamma)$$

$$M_t(\gamma) = e^{-t\Phi(\gamma)} \sum_{v \in \partial \mathcal{T}_t} L(v)^\gamma$$

$$\sum_{v \in \partial \mathcal{T}_t} L(v) X(v)$$

$\gamma \in (0,1) \cup (1,2)$ ($\mathbb{E}[X] = 0$ if $\gamma > 1$)

$$\mathbb{P}[X > t] \sim c^+ t^{-\gamma} \quad \text{and} \quad \mathbb{P}[X < -t] \sim c^- t^{-\gamma}$$

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$$e^{-t\Phi(\gamma)/\gamma} \sum_{v \in \partial \mathcal{T}_t} L(v) X(v)$$

$$= M_t(\gamma)^{1/\gamma} \sum_{v \in \partial \mathcal{T}_t} \frac{L(v)}{\left(\sum_{s \in \partial \mathcal{T}_s} L(v)^{\gamma}\right)^{1/\gamma}} X(v) \to M_\infty^{1/\gamma} \cdot \mathcal{L}_\gamma$$

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$$\mathbb{P}[X > t] \sim c^+ t^{-\gamma} \quad \text{and} \quad \mathbb{P}[X < -t] \sim c^- t^{-\gamma}$$

$$\begin{aligned}\log \varphi_0(\xi) &= \log \mathbb{E}\left[e^{i\xi X}\right] \\ &= -k_0 \xi^\gamma (1 - i\eta_0 \tan(\pi\gamma/2)\mathrm{sign}(\xi)) + o(|\xi|^\gamma) \\ &= \log g(\xi) + o(|\xi|^\gamma).\end{aligned}$$

$$k_0 = \frac{(c^+ + c^-)\pi}{2\Gamma(\gamma)\sin(\pi\gamma/2)}, \qquad \eta_0 = \frac{c^+ - c^-}{c^+ + c^-}$$

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$$\mathbb{E}\left[\exp\left\{\sum_{v \in \partial\mathcal{T}_t} i\xi L(v) X(v)\right\}\right]$$

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$$\sum_{v \in \mathcal{T}_t} L(v)C(v)$$

$$\sum_{v\in\mathcal{T}_t} L(v)C(v)$$

$(\Pi_n)_{n\in\mathbb{N}}$ - multiplicative random walk, $(N_t)_t$ - homogeneous Poisson process

$$\mathbb{E}\left[\sum_{v\in\mathcal{T}_t} g(C(v),L(v))\right] = \mathbb{E}\left[\sum_{k=0}^{N_t-1} M_k^{-\alpha} e^{\lambda(\alpha)k} g(M_k,C)\right]$$

$$\sum_{v \in \mathcal{T}_t} L(v) C(v)$$

$(\Pi_n)_{n \in \mathbb{N}}$ - multiplicative random walk, $(N_t)_t$ - homogeneous Poisson process

$$\mathbb{E}\left[\sum_{v \in \mathcal{T}_t} g(C(v), L(v))\right] = \mathbb{E}\left[\sum_{k=0}^{N_t - 1} M_k^{-\alpha} e^{\lambda(\alpha) k} g(M_k, C)\right]$$

In particular if $\mathbb{E}[C] = 0$,

$$\mathbb{E}\left[\left(\sum_{v \in \mathcal{T}_t} L(v) C(v)\right)^2\right] = \mathbb{E}\left[\sum_{v \in \mathcal{T}_t} L(v)^2 C(v)^2\right] = \mathbb{E}[C^2] \frac{e^{t\Phi(2)} - 1}{\Phi(2)}$$

$$e^{-t\Phi(2)/2} \sum_{v \in \mathcal{T}_t} L(v) C(v)$$

$$
\begin{aligned}
e^{-t\Phi(2)/2} &\sum_{v \in \mathcal{T}_t} L(v)C(v) \\
&= e^{-t\Phi(2)/2} \sum_{v \in \mathcal{T}_s} L(v)C(v) + e^{-t\Phi(2)/2} \sum_{u \in \partial\mathcal{T}_s} \sum_{u \leq v \in \mathcal{T}_t} L(v)C(v)
\end{aligned}
$$

$$
\begin{aligned}
e^{-t\Phi(2)/2} & \sum_{v\in\mathcal{T}_t} L(v)C(v) \\
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&= o(1) + e^{-t\Phi(2)/2} \sum_{u\in\partial\mathcal{T}_s} L(u) \sum_{u\leq v\in\mathcal{T}_t} \frac{L(v)}{L(u)} C(v)
\end{aligned}
$$

$$
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e^{-t\Phi(2)/2} &\sum_{v \in \mathcal{T}_t} L(v) C(v) \\
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&= o(1) + e^{-t\Phi(2)/2} \sum_{u \in \partial \mathcal{T}_s} L(u) \sum_{u \leq v \in \mathcal{T}_t} \frac{L(v)}{L(u)} C(v) \\
&= (1 + o(1)) \mathcal{N} \left(0, e^{-s\Phi(2)} \sum_{u \in \partial \mathcal{T}_s} L(u)^2 \mathbb{E}[C^2] / \Phi(2) \right)
\end{aligned}
$$

$$
\begin{aligned}
e^{-t\Phi(2)/2} & \sum_{v\in\mathcal{T}_t} L(v)C(v) \\
&= e^{-t\Phi(2)/2} \sum_{v\in\mathcal{T}_s} L(v)C(v) + e^{-t\Phi(2)/2} \sum_{u\in\partial\mathcal{T}_s} \sum_{u\le v\in\mathcal{T}_t} L(v)C(v) \\
&= o(1) + e^{-t\Phi(2)/2} \sum_{u\in\partial\mathcal{T}_s} L(u) \sum_{u\le v\in\mathcal{T}_t} \frac{L(v)}{L(u)} C(v) \\
&= (1+o(1))\mathcal{N}\left(0, e^{-s\Phi(2)} \sum_{u\in\partial\mathcal{T}_s} L(u)^2 \mathbb{E}[C^2]/\Phi(2)\right) \\
&= (1+o(1))\mathcal{N}\left(0, M_\infty(2)\mathbb{E}[C^2]/\Phi(2)\right)
\end{aligned}
$$

Theorem (Buraczewski, D, Marynych, 2023)

Consider $\varphi_0(\xi) = \mathbb{E}[e^{i\xi X}]$, $A_1, A_2 > 0$

$$\partial_t \varphi_t(\xi) + \varphi_t(\xi) = \widehat{Q_+}(\varphi_t, \varphi_t)(\xi) = \mathbb{E}\left[\varphi_t(A_1\xi)\varphi_t(A_2\xi)e^{i\xi C}\right]$$

Then

- $\mathbb{P}[|X| > t] \sim (c_+ + c_-)t^{-\gamma}$, $\mathbb{E}[|C|^{1+\delta}] < \infty$, $\Phi(1) > 0$, $\Phi'(1) - \Phi(1) < 0$

$$\varphi_t(e^{-\Phi(\gamma)t/\gamma}\xi) \to \mathbb{E}\left[e^{-k_0 L(v)^\gamma M_\infty(\gamma)\xi^\gamma(1 - i\eta_0 \tan(\pi\gamma/2)\mathrm{sign}(\xi))}\right]$$

- $\mathbb{E}[C] = \mathbb{E}[X] = 0$, $\mathbb{E}[C^2], \mathbb{E}[X^2] < \infty$, $\Phi(2) > 0$, $2\Phi'(2) - \Phi(2) < 0$

$$\varphi_t(e^{-\mu(2)t/2}\xi) \to \mathbb{E}\left[\exp\left\{-\xi^2 M_\infty(2)(\mathbb{E}[X^2] + \mathbb{E}[C^2]/\Phi(2))/2\right\}\right]$$

- $\mathbb{E}[|C|^{1+\delta}], \mathbb{E}[|X|^{1+\delta}] < \infty$, $\Phi(1) < 0$, $\Phi'(1) - \Phi(1) < 0$

$$\varphi_t(\xi) \to \mathbb{E}[e^{i\xi C_\infty}], \qquad C_\infty \overset{d}{=} A_1 C_\infty^{(1)} + A_2 C_\infty^{(2)} + C$$