Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 127 (2017) 3014–3041

stochastic processes and their applications

www.elsevier.com/locate/spa

Thin tails of fixed points of the nonhomogeneous smoothing transform

Gerold Alsmeyer[a], Piotr Dyszewski[b,*]

[a] *Inst. Math. Statistics, Department of Mathematics and Computer Science, University of Münster, Orléans-Ring 10, D-48149 Münster, Germany*

[b] *Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland*

Received 26 October 2015; received in revised form 17 January 2017; accepted 24 January 2017
Available online 7 February 2017

Abstract

For a given random sequence $(C, T_1, T_2, \ldots)$, the smoothing transform $\mathcal{S}$ maps the law of a real random variable X to the law of $\sum_{k\geq 1} T_k X_k + C$, where $X_1, X_2, \ldots$ are independent copies of X and also independent of $(C, T_1, T_2, \ldots)$. This law is a fixed point of $\mathcal{S}$ if $X \stackrel{d}{=} \sum_{k\geq 1} T_k X_k + C$ holds true, where $\stackrel{d}{=}$ denotes equality in law. Under suitable conditions including $\mathbb{E}C = 0$, $\mathcal{S}$ possesses a unique fixed point within the class of centered distributions, called the canonical solution because it can be obtained as a certain martingale limit in an associated weighted branching model. The present work provides conditions on $(C, T_1, T_2, \ldots)$ such that the canonical solution exhibits right and/or left Poissonian tails and the abscissa of convergence of its moment generating function can be determined. As a particular application, the right tail behavior of the Quicksort distribution is found.

MSC: 60H25 (60E10)

Keywords: Nonhomogeneous smoothing transform; Stochastic fixed point; Moment generating function; Exponential moment; Poissonian tail; Weighted branching process; Forward and backward equation; Quicksort distribution

* Corresponding author.
E-mail addresses: gerolda@math.uni-muenster.de (G. Alsmeyer), piotr.dyszewski@math.uni.wroc.pl (P. Dyszewski).

http://dx.doi.org/10.1016/j.spa.2017.01.008
0304-4149/© 2017 Published by Elsevier B.V.

1. Introduction

Given a sequence $(C, T_1, T_2, \ldots)$ of real-valued random variables with an a.s. finite number $N := \sum_{k\geq 1} \mathbf{1}_{\{T_k \neq 0\}}$ of nonzero T_k and a nonzero random variable C, we consider the associated nonhomogeneous smoothing transform

$$\mathcal{S} : F \;\mapsto\; \mathcal{L}\left(\sum_{k\geq 1} T_k X_k + C\right)$$

which maps a distribution F on $\mathbb{R}$ to the law of $\sum_{k\geq 1} T_k X_k + C$, where $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) with common law F and independent of $(C, T_1, T_2, \ldots)$. In the case when $\mathcal{S}F = F$, the distribution F is called a fixed point of $\mathcal{S}$. In terms of random variables, this may be stated as a so-called *stochastic fixed-point equation (SFPE)*, namely

$$X \;\stackrel{d}{=}\; \sum_{k\geq 1} T_k X_k + C, \tag{1}$$

where X is a copy of $X_1, X_2, \ldots$ and $\stackrel{d}{=}$ means equality in law.

Special instances of (1) appear above all in the asymptotic analysis of objects that exhibit a certain kind of self-similar random recursive structure like random trees, branching processes, or recursive algorithms and data structures [1,9,18], but also in stochastic geometry [19,20]. As a particularly prominent and by now classic example, we mention `Quicksort`, a recursive divide-and-conquer algorithm to sort a list of n distinct numbers. Rösler [21] has shown that, as $n \to \infty$, the suitably normalized number of key comparisons needed by `Quicksort` when applied to a random permutation of n numbers converges in distribution to a random variable $X^{(qs)}$ which satisfies the SFPE

$$X^{(qs)} \;\stackrel{d}{=}\; UX_1^{(qs)} + (1-U)X_2^{(qs)} + g(U), \tag{2}$$

called *Quicksort equation*. Here U has a uniform distribution on $(0, 1)$ and $g : [0, 1] \to \mathbb{R}$ denotes an explicitly given continuous function. For further information regarding this result, the algorithm itself and an application of our results to it, the reader is referred to Section 5. There we will also treat two relatives of Eq. (2), viz. the *median-of-three version of* `Quicksort` *equation*, again due to Rösler [23], and the *2-dimensional quad tree equation* obtained by Neininger and Rüschendorf [17] for the normalized asymptotic internal path length of random quad trees.

In the case $N = 1$, which is *not* our focus here, the SFPE (1) takes the simple form

$$X \;\stackrel{d}{=}\; T_1 X + C, \tag{3}$$

called *random difference equation*. Random variables satisfying this equation, called *perpetuities* due to a special interpretation in the context of Mathematical Finance, appear in various quite different areas like number theory, combinatorics, branching processes in random environment, or additive-increase multiplicative-decrease (AIMD) algorithms [11].

The principal aim of this work is to provide conditions on $(C, T_1, T_2, \ldots)$ such that the solutions to (1) exhibit thin tails in the sense that they possess finite exponential moments. More precisely, we will study the domain of the moment generating function (mgf) of a random variable X solving (1), thus

$$\{\theta \in \mathbb{R} : \mathbb{E}e^{\theta X} < \infty\},$$

and in fact give a precise description of this set in some special cases including random difference equations. Regarding the latter, this will answer a question posed in [4, Section 4] about the abscissa of convergence of the m.g.f. of X (see results in Section 3), i.e.

$$r^*(X) := \sup\{\theta \in \mathbb{R} : \mathbb{E}e^{\theta X} < \infty\}$$

and

$$r_*(X) := \inf\{\theta \in \mathbb{R} : \mathbb{E}e^{\theta X} < \infty\}.$$

Since, by Lemma 3.6 in [8],

$$r^*(X) := \limsup_{x\to\infty} \frac{-\log \mathbb{P}[X > x]}{x}$$

and

$$r_*(X) := \liminf_{x\to\infty} \frac{\log \mathbb{P}[X < -x]}{x}$$

this gives also information about the rate of exponential decay of the right and left tail of X.

Building on observations made by Goldie and Grübel [10] and later Hitczenko and Wesołowski [12] in the case $N = 1$, we will also investigate the relation between the tail of X and $\max_{1\le k\le N} T_k$. More precisely, we will show that X exhibits Poissonian tails, viz.

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[X > x]}{x \log x} = -\frac{\gamma}{\|C^+\|_\infty}$$

provided that C is a.s. bounded, $T_1, T_2, \ldots$ are nonnegative,

$$\left\| \sum_{k=1}^N T_k^p \right\|_\infty \le 1 \quad \text{and} \quad \mathbb{P}\left[\max_{1\le k\le N} T_k \in (1-\varepsilon, 1] \middle| C > c \right] \overset{\varepsilon\downarrow 0}{\asymp} \varepsilon^\gamma$$

for some $p, \gamma > 0$ and all $c < \|C^+\|_\infty$. Here and throughout $\|\cdot\|_r$ denotes the usual L^r-norm for $r \in [1,\infty]$ and $f(\varepsilon) \overset{\varepsilon\downarrow 0}{\asymp} g(\varepsilon)$ means that

$$0 < \liminf_{\varepsilon\downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} \le \limsup_{\varepsilon\downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty.$$

We have organized the paper as follows. Section 2 introduces some basic notation and assumptions including the definition of infinite-order Ulam–Harris trees and weighted branching processes. Our main results regarding exponential moments are stated in Section 3, followed by Section 4 containing our Poissonian tail result (Theorem 4.1) and more information about how it relates to the results in [10,12]. A discussion of the examples mentioned above in the light of Theorem 4.1 can be found in Section 5, while the proofs of our main results are provided in Section 6.

2. Preliminaries

Let (the law of) X be a solution to (1). Due to the independence of $(C, T_1, T_2, \ldots)$ and $X_1, X_2, \ldots$ on the right-hand side the SFPE remains valid under any measurable rearrangement of the nonzero T_k in $(C, T_1, T_2, \ldots)$. Since N is a.s. finite we may therefore assume without loss of generality that

$$T_1 \ge T_2 \ge \cdots \ge T_N \quad \text{and} \quad T_{N+1} = T_{N+2} = \cdots = 0,$$

so that (1) becomes

$$X \stackrel{d}{=} \sum_{k=1}^{N} T_k X_k + C. \tag{4}$$

As we are dealing with integrable solutions X, a replacement of X, X_k and C in (4) with their centerings

$$X^0 := X - \mathbb{E}X, \quad X_k^0 := X_k - \mathbb{E}X_k \quad \text{and} \quad C^0 := C - \mathbb{E}X - \mathbb{E}\left[X\sum_{k=1}^{N} T_k\right]$$

respectively, leads to a SFPE of the same type, namely

$$X^0 \stackrel{d}{=} \sum_{k=1}^{N} T_k X_k^0 + C^0.$$

Hence we may w.l.o.g. assume for the rest of this article that

$$\mathbb{E}X = \mathbb{E}C = 0.$$

The power sums

$$\Sigma_\alpha := \sum_{k=1}^{N} |T_k|^\alpha$$

for $\alpha \in \mathbb{R}_\geqslant = [0, \infty)$ will play an important role in our analysis, where $\Sigma_0 = N$.

The following weighted branching model is needed to describe the iterations of the SFPE (4). Consider an infinite-order Ulam–Harris tree with vertex set $\mathbb{T} = \{\varnothing\} \cup \bigcup_{n\geq 1} \mathbb{N}^n$ of finite integer words and with the empty word $\varnothing$ being its root. We will write $v_1 \ldots v_n$ as shorthand for $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$ and put $v_{|k} := v_1 \ldots v_k$ for $k = 1, \ldots, n$ and $v_{|0} := \varnothing$. For any two finite words $v, w \in \mathbb{T}$ let $vw = (v, w)$ be its juxtaposition. $\mathbb{T}$ is given the tree structure by placing an edge between v and vi for any $v \in \mathbb{T}$ and $i \in \mathbb{N}$. Next, let $(C(v), T_1(v), T_2(v), \ldots)$ be i.i.d. copies of $(C, T_1, T_2, \ldots)$. Interpret $T_i(v)$ as a weight attached to the edge connecting v and vi and $C(v)$ as a weight attached to vertex v. Define $L(v)$ as the total weight of the branch from the root to $v = v_1 \ldots v_n$, obtained by multiplication of the edge weights along this path, thus

$$L(v) := \prod_{k=0}^{n-1} T_{v_{k+1}}(v_{|k}),$$

where $L(\varnothing) := 1$. Then put

$$\Sigma_{p,n} := \sum_{|v|=n} |L(v)|^p$$

for $n \in \mathbb{N}$, $p \in \mathbb{R}_\geqslant$ and note that $\|\Sigma_{p,n}\|_1 = \|\Sigma_p\|_1^n$ for all $n \in \mathbb{N}$. Finally, let

$$N_n := \#\{v : |v| = n \text{ and } L(v) > 0\} = \Sigma_{0,n}$$

denote the number of branches of length n with positive weight. Obviously, $N_1 \stackrel{d}{=} N$, and with $N_0 = 1$, the sequence $(N_n)_{n\geq 0}$ forms a simple Galton–Watson process.

Next, with $X(v)$ denoting i.i.d. copies of X which are independent of all other occurring random variables, n-fold iteration of (4) provides us with

$$X \stackrel{d}{=} \sum_{|v|=n} L(v)X(v) + \sum_{k=0}^{n-1}\sum_{|v|=k} L(v)C(v) \tag{5}$$

for all $n \geq 1$. If the second term on the right-hand side of (5), that is $W_{n-1} := \sum_{k=0}^{n-1}\sum_{|v|=k} L(v)C(v)$, converges a.s. to

$$W := \sum_{k\geq 0}\sum_{|v|=k} L(v)C(v),$$

then W is also a solution to the SFPE (4), called *canonical solution*. Notice that

$$W_n \stackrel{d}{=} \mathcal{S}^{n+1}(\delta_0)$$

for each $n \in \mathbb{N}$, where δ_0 denotes the Dirac measure at 0. The recursive structure provides us with two useful equations for the W_n, the first of which being

$$W_n = W_{n-1} + \sum_{|v|=n} L(v)C(v), \tag{6}$$

called *forward equation*, which is just a consequence of the definition of W_n. Since $W_n \stackrel{d}{=} \mathcal{SL}(W_{n-1})$ we also have

$$W_n = \sum_{k=1}^{N(\varnothing)} T_k(\varnothing)W_{n-1}(k) + C(\varnothing), \tag{7}$$

called *backward equation*, where $W_{n-1}(1), W_{n-1}(2), \ldots$ are independent copies of W_{n-1} and also independent of $(C(\varnothing), T_1(\varnothing), T_2(\varnothing), \ldots)$. We further point out that, given $p \in [1, 2]$, $(W_n)_{n\geq 0}$ forms a L^p-convergent, zero mean martingale if

$$\mathbb{E}C = 0, \qquad \|C\|_p < \infty \quad \text{and} \quad \|\Sigma_p\|_1 < 1. \tag{8}$$

The martingale property is easily verified, and the L^p-convergence follows from

$$\begin{aligned}
\|W_{m+n} - W_m\|_p &\leq \sum_{k=m+1}^{m+n} \left\| \sum_{|v|=k} L(v)C(v) \right\|_p \\
&\leq 2^{1/p}\|C\|_p \sum_{k=m+1}^{m+n} \|\Sigma_{p,k}\|_1 \\
&\leq 2^{1/p}\|C\|_p \frac{\|\Sigma_p\|_1^{m+1}}{1-\|\Sigma_p\|_1},
\end{aligned}$$

for all $m, n \geq 1$, where for the second line we have used

- the *double martingale structure* of $(W_n)_{n\geq 0}$, first systematically utilized in [7] for the study of moments of the ordinary Galton–Watson process and later in [5] for general weighted branching processes,

- the Topchiĭ–Vatutin inequality for martingales as stated in [6], here applied to $\left\|\sum_{|v|=k} L(v)C(v)\right\|_p$.

The double martingale structure refers to the fact that each increment of $(W_n)_{n\geq 0}$, viz.

$$W_n - W_{n-1} = \sum_{|v|=n} L(v)C(v) \quad (n \geq 1),$$

forms itself a martingale sum when conditioned upon $\sigma(L(v),\ |v| \leq n)$.

As a particular consequence of (8), $(W_n)_{n\geq 0}$ is uniformly integrable and thus a Doob martingale, i.e. $W_n = \mathbb{E}(W|\mathcal{F}_n)$ for each $n \geq 0$, where $\mathcal{F}_n = \sigma(W_0, \ldots, W_n)$. If $p = 2$, we further have

$$\|W_n\|_2^2 = \|W_{n-1}\|_2^2 + \|\Sigma_{2,n}\|_1 \|C\|_2^2 = \|W_{n-1}\|_2^2 + \|\Sigma_2\|_1^n \|C\|_2^2$$

for each $n \geq 1$, giving

$$\sigma_W^2 := \|W\|_2^2 = \sup_{n\geq 0} \|W_n\|_2^2 = \frac{\|C\|_2^2}{1 - \|\Sigma_2\|_1} = \frac{\mathbb{E}C^2}{1 - \mathbb{E}\Sigma_2}.$$

Let us finally point out that the (law of the) canonical solution W is in fact the unique zero-mean fixed point of $\mathcal{S}$ in L^p, see Rösler [22, Thm. 3] for the case $p = 2$ and [2, Thm. 1 and Thm. 3] for general $p \in [1, 2]$.

We proceed with the introduction of some further notation. The mgf's of C, W_n and W are denoted by φ, Ψ_n and Ψ with canonical domains $\mathbb{D}_\varphi$, $\mathbb{D}_{\Psi_n}$ and $\mathbb{D}_\Psi$, respectively, thus

$$\mathbb{D}_\varphi := \{\theta \in \mathbb{R} : \mathbb{E}e^{\theta C} < \infty\}, \quad \text{etc.}$$

We close this section with a basic lemma and note beforehand that, if $\mathbb{D}_\varphi \neq \{0\}$, we may assume w.l.o.g. that $\mathbb{D}_\varphi \cap \mathbb{R}_\geqslant \neq \{0\}$, for otherwise the latter holds after switching from C to $-C$ (and thus from W to $-W$).

Lemma 2.1. *Suppose that* (8) *holds for some* $p \in [1, 2]$. *Then*

(a) $\varphi(\theta) = \Psi_0(\theta) \leq \Psi_1(\theta) \leq \cdots \leq \Psi(\theta)$ *for all* $\theta \in \mathbb{R}$ *and thus* $\mathbb{D}_\Psi \subset \mathbb{D}_\varphi$.

(b) $\Psi(\theta) = \lim_{n\to\infty} \Psi_n(\theta)$ *for all* $\theta \in \mathbb{R}$.

Proof. As shown above, condition (8) for some $p \in [1, 2]$ ensures that $(W_n)_{n\geq 0}$ is a zero-mean Doob martingale, in particular $\mathbb{E}W = 0$. Consequently, Ψ is convex on its domain with unique minimum at 0. Moreover, by using Jensen's inequality and $W_0 = C(\varnothing)$,

$$\Psi(\theta) = \mathbb{E}e^{\theta W} = \mathbb{E}\left[\mathbb{E}(e^{\theta W}|\mathcal{F}_n)\right] \geq \mathbb{E}e^{\theta\mathbb{E}(W|\mathcal{F}_n)} = \mathbb{E}e^{\theta W_n} = \Psi_n(\theta),$$

for all $\theta \geq 0$ and $n \geq 0$. Similarly, $\Psi_n(\theta) \geq \Psi_{n-1}(\theta)$ can be shown, which is in fact a trivial consequence of the submartingale property of $(e^{\theta W_n})_{n\geq 0}$ if the latter sequence is integrable. We have thus proved (a), and (b) then follows because, by an appeal to Fatou's lemma, we also have

$$\Psi(\theta) = \mathbb{E} \lim_{n\to\infty} e^{\theta W_n} \leq \lim_{n\to\infty} \mathbb{E}e^{\theta W_n} = \lim_{n\to\infty} \Psi_n(\theta) \quad \text{for all } \theta \in \mathbb{R}. \quad \square$$

3. Exponential moments

The most natural approach to study Ψ is via the functional equation it satisfies as a consequence of the SFPE (4). Namely, writing the latter in terms of m.g.f.'s and conditioning upon $(C, T_1, T_2, \dots)$ leads to

$$\Psi(\theta) = \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Psi(T_k\theta)\right] \quad \text{for } \theta \in \mathbb{D}_\Psi. \tag{9}$$

Aiming to determine $\mathbb{D}_\Psi$, Theorem 3.2 constitutes a good starting point because it allows us to focus thereafter on the situation when

$$\max_{1\le k\le N} |T_k| \le 1 \quad \text{a.s.} \tag{10}$$

However, we first state the following basic result, in essence due to Goldie and Grübel [10], about the situation when (10) fails.

Proposition 3.1. *If (8) holds for some $p \in [1, 2]$ and*

$$\mathbb{P}\left[\max_{1\le k\le N} |T_k| > 1\right] > 0,$$

then

$$\mathbb{E}e^{\theta|W|} = \infty$$

for all $\theta \in \mathbb{R}\setminus\{0\}$, thus $\mathbb{D}_\Psi \cap \mathbb{R}_\geqslant = \{0\}$ or $\mathbb{D}_\Psi \cap \mathbb{R}_\leqslant = \{0\}$.

Proof. Since the $T_1 \ge \dots \ge T_N$, we have $\mathbb{P}(T_1 > 1) > 0$ or $\mathbb{P}(T_N < -1) > 0$. Putting $B_1 := \sum_{k=2}^{N} T_k W_k + C$ and $B_N := \sum_{k=1}^{N-1} T_k W_k + C$, observe that W satisfies the random difference equations

$$W \stackrel{d}{=} T_1 W_1 + B_1 \quad \text{and} \quad W \stackrel{d}{=} T_N W_N + B_N$$

with (B_1, T_1), (B_N, T_N) being independent of W_1 and W_N, respectively. Now use Theorem 4.1 in [10] to infer that

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}[|W| > t]}{\log t} > -\infty$$

and thus in particular $\mathbb{E}e^{\theta|W|} = \infty$ for all $\theta \in \mathbb{R}$. □

Having shown that $\mathbb{D}_\Psi$ cannot contain an open neighborhood of 0 if (10) fails, Theorem 3.2 contains more detailed information for this situation and particularly reveals that $\mathbb{D}_\Psi$ differs for the cases when the T_k are $[-1, 1]$-valued or $[0, 1]$-valued, more precisely, if

$$\beta := \|T_N^-\|_\infty = \operatorname{ess\,sup} T_N^- \tag{11}$$

is 0, thus $\mathbb{P}[T_k > 0, k \le N] = 1$, or > 0, thus $\mathbb{P}[T_k > 0, k \le N] < 1$, where $T_N = \min_{1\le k\le N} T_k$ should be recalled.

Theorem 3.2. *Suppose that (8) holds for some $p \in [1, 2]$ and*

$$\|C\|_2 < \infty \quad \textit{and} \quad \mathbb{E}e^{\theta\Sigma_2} < \infty \quad \textit{for some } \theta > 0. \tag{12}$$

Then

$$\mathbb{D}_{\Psi} \neq \{0\},$$

if and only if one of the following four cases occurs:

(a1) (10) *holds,* $\beta = 0$*, and* $\mathbb{D}_{\varphi} \neq \{0\}$.
(a2) (10) *holds,* $\beta > 0$*, and* $\mathbb{D}_{\varphi} \supset (-\varepsilon, \varepsilon)$ *for some* $\varepsilon > 0$.
(a3) $\mathbb{P}[T_1 > 1] > 0$*,* $\beta = 0$*, and* $\mathbb{P}\left[w^* \Sigma_1 + C \leq w^*\right] = 1$ *for some* $w^* \geq 0$.
(a4) $\mathbb{P}[T_1 > 1] > 0$*,* $\beta = 0$*, and* $\mathbb{P}\left[w_* \Sigma_1 + C \geq w_*\right] = 1$ *for some* $w_* \leq 0$.

Remark 3.3. Our proof will actually provide a slightly stronger result, stated as Theorem 6.2 in the proof section. The reader should notice that Assumption (12) particularly holds if $\|\Sigma_2\|_\infty < \infty$ or, a fortiori, if (10) and $\|N\|_\infty < \infty$ are valid. Moreover, W has no nontrivial exponential moments whenever $\beta > 1$, i.e. $\mathbb{P}[T_N < -1] > 0$.

In view of the previous result, we will focus hereafter on the situation when (10) holds. As we will see, it is the behavior of the T_k's near their extremal values 1 and $-\beta$ which determines $\mathbb{D}_{\Psi}$. The key condition which allows us to provide a transparent necessary and sufficient criterion for $\Psi(\theta) < \infty$ is that only one of the weights may take values arbitrarily close to 1 or $-\beta$ with positive probability. Informally speaking, we will assume in the case $\beta = 0$ that

$$\#\{j \in \{1, \ldots, N\} : T_j \approx 1\} \leq 1 \quad \text{a.s.} \tag{13}$$

which in some sense entails $T_2 \ll 1$, and in the case $\beta > 0$ that

$$\#\{j \in \{1, \ldots, N\} : T_j \approx 1 \text{ or } T_j \approx -\beta\} \leq 1 \quad \text{a.s.}$$

Theorem 3.4 provides a very simple explicit description of $\mathbb{D}_{\Psi}$ when $\beta = 0$ (nonnegative T_k's), while Theorem 3.5 deals with the case $\beta > 0$ and thus $\mathbb{P}[T_N < 0] > 0$. As indicated by Theorem 3.2, $\beta > 0$ causes some asymmetry regarding $\mathbb{D}_{\Psi}$ which is encoded in β. The proofs will be based on the construction of a certain supersolution (a technique commonly used in the theory of partial differential equations) of the functional equation (9) (see lemma ta 6.1, 6.3 and 6.4). We refer to Example 5.4 for an instance of very simple nonnegative T_k's not satisfying (13) with the result of a rather complicated condition for $\Psi(\theta) < \infty$.

For $\theta \in \mathbb{R}$, we define

$$a(\theta) := \mathbb{E}e^{\theta C}\mathbf{1}_{\{T_1=1\}} \quad \text{and} \quad b(\theta) := \mathbb{E}e^{\theta C}\mathbf{1}_{\{T_N=-\beta\}}. \tag{14}$$

Also, let $int(A)$ denote the interior of the set $A \subset \mathbb{R}$.

Theorem 3.4. *Suppose* (8) *for some* $p \in [1, 2]$*,* (10)*,* $\beta = 0$ *and* $\|\Sigma_2\|_\infty < \infty$*. Suppose further that*

$$\left\|\max_{2\leq k\leq N} T_k\right\|_\infty < 1 \quad \textit{and} \quad \mathbb{P}[T_1 = 1, N \geq 2] = 0. \tag{15}$$

Then

$$\mathbb{D}_{\Psi} = \mathbb{D}_{\varphi} \cap \{\theta \in \mathbb{R} : a(\theta) < 1\}.$$

Notice that (15) and $\beta = 0$ entail

$$T_2, T_3, \ldots, T_N \leq 1 - \delta \quad \text{a.s.}$$

for all sufficiently small $\delta > 0$, which formalizes (13). To ensure the latter when $\beta > 0$, a more complicated version of (15) must be imposed and appears as (16)–(18) in the subsequent result.

Theorem 3.5. *Suppose (8) for some $p \in [1, 2]$, (10), $\beta > 0$ and $\|\Sigma_2\|_\infty < \infty$. Suppose further that, for some $\delta \in (0, 1)$,*

$$\mathbb{P}[-\beta(1-\delta) \leq T_k \leq 1-\delta,\ 2 \leq k \leq N-1] = 1. \tag{16}$$

$$\mathbb{P}[T_1 > 1-\delta,\ T_N < -\beta(1-\delta)] = 0. \tag{17}$$

$$\mathbb{P}[T_1 = 1, N \geq 2] = \mathbb{P}[T_N = -\beta, N \geq 2] = 0. \tag{18}$$

Then the following assertions hold true:

(a) *If $\mathbb{P}[T_N = -\beta] > 0$ and $\beta < 1$, then*

$$\mathbb{D}_\Psi = \left\{\theta : a(-\beta\theta) \vee a(\theta) < 1 \textit{ and } -\beta\theta, \theta \in \mathbb{D}_\varphi\right\}.$$

(b) *If $\mathbb{P}[T_N = -1] > 0$, then*

$$\mathbb{D}_\Psi = \left\{\theta : (1 - a(-\theta))(1 - a(\theta)) > b(\theta)b(-\theta) \textit{ and } -\theta, \theta \in \mathbb{D}_\varphi\right\}.$$

(c) *If $\mathbb{P}[T_N = -\beta] = 0$, then*

$$\mathit{int}(\mathbb{D}_\Psi) \subset \left\{\theta : a(-\beta\theta) \vee a(\theta) < 1 \textit{ and } -\beta\theta, \theta \in \mathbb{D}_\varphi\right\} \subset \mathbb{D}_\Psi.$$

In the random difference case when $N = 1$ and thus $T_1 = T_N$, it is now easy to derive the abscissa of convergence of Ψ, viz $r_*(W) = \inf \mathbb{D}_\Psi$ and $r^*(W) = \sup \mathbb{D}_\Psi$, from the previous theorems. The details can be left to the reader.

4. Poissonian tails

As shown by Theorem 3.4, the canonical solution W exhibits very thin tails in the sense of possessing exponential moments of any order ($\mathbb{D}_\Psi = \mathbb{R}$) if (8) for some $p \in [1, 2]$, (10), $\beta = 0$, $\|\Sigma_2\|_\infty < \infty$, $\mathbb{P}[T_1 = 1] = 0$, and $\mathbb{D}_\varphi = \mathbb{R}$ hold true. It turns out that in this case the tail behavior of W is determined by the behavior of the law of T_1 in a neighborhood of 1, a phenomenon observed for random difference equations by Goldie and Grübel [10] and later by Hitczenko and Wesołowski [12]. Note that this relation is further investigated in the upcoming work by Kołodziejek [16] concerning the random difference equation (3) with $C = 1$. In a proper setting, the phenomenon carries over to the canonical fixed point of the smoothing transform. Regarding the right tail of W, we will work under the additional assumptions (besides those of Theorem 3.4) that C is bounded and the law of T_1, or its conditional law given $C > c$ for any $c \in (0, \|C^+\|_\infty)$, is equivalent to a beta distribution at 1. The first means that, for some $\gamma > 0$,

$$\exists\, \varepsilon, d, D > 0 : \forall\, \delta \in (0, \varepsilon) :\ d \leq \frac{\mathbb{P}[1-\delta < T_1 \leq 1]}{\delta^\gamma} \leq D \tag{19}$$

and the second that

$$\forall\, c \in (0, c^+) :\ \exists\, \varepsilon, d', D' > 0 : \forall\, \delta \in (0, \varepsilon) :$$
$$d' \leq \frac{\mathbb{P}[1-\delta < T_1 \leq 1 | C > c]}{\delta^\gamma} \leq D' \tag{20}$$

where $c^+ := \|C^+\|_\infty$. Obviously, (19) entails (20) if C and T_1 are independent. Note that (20) implies $\|C^+\|_\infty = \lim_{\delta \to 0} \|C^+ \mathbf{1}_{\{1-\delta \leq T_1 \leq 1\}}\|_\infty$. The biggest values of C are therefore attained on the set where the biggest values of $T_1 = \max_k T_k$ are attained.

Theorem 4.1. *Given the assumptions of Theorem 3.4, thus $\beta = 0$, suppose further $\|\Sigma_q\|_\infty \le 1$ for some $q \ge 1$, $\|C^+\|_\infty < \infty$, and (19) for some $\gamma > 0$. Then*

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}[W > x]}{x \log x} \le -\frac{\gamma}{c^+},$$

If, furthermore, (20) is valid, then the previous result can be sharpened to

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[W > x]}{x \log x} = -\frac{\gamma}{c^+}.$$

Since $-W$ is the canonical fixed point of the smoothing transform pertaining to $(-C, T_1, T_2, \ldots)$, the corresponding version of the theorem for $\mathbb{P}[W < -x]$ can easily be formulated and requires to replace c^+ with $c^- := \|C^-\|_\infty$, and also C with $-C$ in (20).

In the case of a random difference equation ($N = 1$), Theorem 4.1 improves corresponding results by Goldie and Grübel [10, Theorem 3.1] and Hitczenko and Wesołowski [12, Theorem 4], who both assumed independence of C and T_1. Here dependence is allowed through (20). On the other hand, the first reference also provides a similar result in the case $\beta > 0$, see [10, Theorem 3.2], while the second one further considers other regimes of T_1 near 1 resulting in a different tail behavior of W, see [12, Theorems 5 and 6].

Our arguments for the proof of Theorem 4.1 go along similar lines as in [10], but adapted to the case when branching occurs. More precisely, the upper bound for $\mathbb{P}[W > x]$ is derived by first giving an upper bound for Ψ by a function Φ, i.e.

$$\mathbb{E}\left[e^{\theta C} \prod_{k=1}^{N} \Phi(T_k \theta)\right] \le \Phi(\theta),$$

see Lemma 6.1, and then making use of a Chernoff-type argument. The lower bound is obtained by an iteration of the SFPE (4) along the edges of $\mathbb{T}$ with weights $\max_k T_k(v)$ and a subsequent use of the behavior of $\mathbb{P}[\max_k T_k(v) \in (\delta, 1]]$ so as to derive information on the tail of the smoothing transform at a suitable stopping line.

We note that our result, although providing a general upper bound for $\log \mathbb{P}[W > x]$, does not need to be optimal if C and $(T_1, T_2, \ldots)$ are dependent. Loosely speaking, if such a dependence occurs, the asymptotics depend on the joint behavior of C and T_1 on the event $\{T_1 > 1 - \delta\}$ for small δ, as made precise by condition (20). A prominent example exhibiting such kind of dependence of C and T_1 appears in the `Quicksort` equation (2) for which a discussion can be found in the next section.

5. Examples

Example 5.1. We begin with a discussion of the `Quicksort` distribution in the light of Theorem 4.1. The main result, Eq. (23) on its right tail, has also been obtained in a slightly stronger form by Janson [15] in a recent note. He further proved that its left tail shows a very different behavior in being doubly exponential (Gumbel-like).

`Quicksort`, first introduced by Hoare [13,14], is a widely known so-called *divide-and-conquer algorithm* to sort a list of n distinct real numbers, say $a_1, \ldots, a_n$, in increasing order. Based on the general idea to successively divide a given task into subtasks of the same kind but smaller dimension, it may be briefly described as follows: The first step is to create two sublists by first choosing an element x from the list, called *pivot*, and then to put all a_k smaller than x in the first sublist and all a_k bigger than x in the second sublist. The same procedure is then applied to the two sublists and all further on created ones as long as these contain at least two elements. Hence, the algorithm terminates when all sublists consist of only one element which are then merged to yield $a_1, \dots, a_n$ in increasing order. The way the pivots are chosen throughout the performance of the algorithm may be deterministic or at random, e.g. by picking any element of a given sublist with equal probability. Notice that the particular values of $a_1, \dots, a_n$ do not matter for the algorithm so that we may assume w.l.o.g. that $(a_1, \dots, a_n)$ is a permutation of the numbers $1, \dots, n$. When picking such a permutation at random, the number of key comparisons needed by `Quicksort` to output the ordered sample becomes a random variable Y_n whose expectation and variance are of the order $2n \log n$ and $(7 - \frac{2}{3})\pi^2 n^2$, respectively, as $n \to \infty$. For the case when the pivot is chosen uniformly at random, Rösler [21] showed that the normalization $X_n := n^{-1}(Y_n - \mathbb{E}Y_n)$ converges in distribution to a centered, square-integrable random variable $X^{(qs)}$ satisfying the SFPE (`Quicksort` equation) (2), i.e.

$$X^{(qs)} \stackrel{d}{=} U X_1^{(qs)} + (1-U) X_2^{(qs)} + g(U),$$

where $X_1^{(qs)}, X_2^{(qs)}$ are i.i.d. copies of $X^{(qs)}$ independent of U, a uniformly distributed random variable on $(0, 1)$. Moreover, $g(t) = 2t \log t + 2(1-t) \log(1-t) + 1$ for $t \in (0, 1)$.

Obviously, this SFPE fits into our framework with $N = 2$, $T_1 = U \vee (1-U)$, $T_2 = U \wedge (1-U)$ and

$$C = g(U) = 2\, T_1 \log T_1 + 2\, (1 - T_1) \log(1 - T_1) + 1.$$

Note also that $\Sigma_1 = 1, \mathbb{P}[T_1 > 1 - \delta] = 2\delta$ for $\delta \in (0, 1/2)$, $\|C^+\|_\infty = 1$ and $\|C^-\|_\infty = 2 \log 2 - 1$, where the last two facts follow because

$$\begin{aligned} \|C^+\|_\infty &= \sup_{t \in (0,1)} g(t) = \lim_{t \uparrow 1} g(t) = 1, \\ \|C^-\|_\infty &= - \inf_{t \in (0,1)} g(t) = -g(1/2) = 2 \log 2 - 1. \end{aligned}$$

The first part of Theorem 4.1 therefore provides us with

$$\limsup_{x \to \infty} \frac{\log \mathbb{P}[X^{(qs)} > x]}{x \log x} \le -1 \quad \text{and} \tag{21}$$

$$\limsup_{x \to \infty} \frac{\log \mathbb{P}[X^{(qs)} < -x]}{x \log x} \le - \frac{1}{2 \log 2 - 1} \approx -2.5887. \tag{22}$$

Regarding (20), we have that, for any $c \in (0, 1)$ and $t \in (1 - \eta_c, 1)$,

$$\mathbb{P}[T_1 > 1 - t | C > c] = \frac{\mathbb{P}[U \notin (t, 1-t)]}{\mathbb{P}[U \notin (\eta_c, 1 - \eta_c)]} = \frac{t}{\eta_c},$$

where η_c is the unique value in $[0, \frac{1}{2})$ such that $g(\eta_c) = c$. Consequently, (21) can be sharpened to

$$\lim_{x \to \infty} \frac{\log \mathbb{P}[X^{(qs)} > x]}{x \log x} = -1. \tag{23}$$

As already mentioned, the behavior of $\log \mathbb{P}[X^{(qs)} < -x]$ is very different and therefore (20) must be violated. Indeed, $-C = -g(U)$ attains its maximal values when U is close to $\frac{1}{2}$. As a consequence, $\{T_1 > 1 - t\}$ and $\{-C > c\}$ are disjoint and hence

$$\mathbb{P}[T_1 > 1 - t | - C > c] = 0$$

for all c close to c^- and t sufficiently close to 1.

Example 5.2. A similar analysis can be done for the median-of-three version of `Quicksort` when the pivot is chosen as the median of three randomly chosen elements of the list. This makes for a more balanced partitioning at the cost of computing the median. The corresponding SFPE, called *median-of-three Quicksort equation*, has again been derived by Rösler [23] and is of the form

$$X^{(mtqs)} \stackrel{d}{=} MX_1^{(mtqs)} + (1-M)X_2^{(mtqs)} + f(M),$$

with $f(t) := 1 + \frac{12}{7}(t\log t + (1-t)\log(1-t))$ for $t \in (0,1)$ and $M = med(U_1, U_2, U_3)$ for independent uniform $(0,1)$ variables $U_i, i = 1,2,3$. The latter appears because the median-of-three version of `Quicksort` chooses the partitioning element (pivot) of a sublist in each division step as the median of a small random sample (here of size 3). Noting that M has a $\beta(1,1)$ distribution with density $6x(1-x)\mathbf{1}_{(0,1)}(x)$ and that $T_1 = M \vee (1-M)$ satisfies

$$\mathbb{P}[1-\delta \le T_1 \le 1] = \mathbb{P}[M \le \delta] + \mathbb{P}[M \ge 1-\delta] = 6\delta^2 - 3\delta^3$$

for $0 < \delta < 1/2$, we find by the same arguments as in Example 5.1 that

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[X^{(mtqs)} > x]}{x\log x} = -2 \quad \text{and} \tag{24}$$

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}[X^{(mtqs)} < -x]}{x\log x} \le -\frac{14}{12\log 2 - 7} \approx -10.624. \tag{25}$$

We thus see that the right tail for the normalized number of key comparisons is asymptotically thinner for the median-of-three version of `Quicksort` than for its standard counterpart.

Example 5.3. The last example mentioned in the Introduction is the following equation obtained by Neininger and Rüschendorf [17] for the asymptotic normalized internal path length in a 2-dimensional quad tree, viz.

$$\begin{aligned} X^{(qt)} \stackrel{d}{=} {} & U_1U_2X_1^{(qt)} + U_1(1-U_2)X_2^{(qt)} + (1-U_1)U_2X_3^{(qt)} \\ & + (1-U_1)(1-U_2)X_4^{(qt)} + h(U_1,U_2), \end{aligned}$$

where $h : (0,1)^2 \to \mathbb{R}$ is defined as

$$\begin{aligned} h(u_1,u_2) = {} & 1 + u_1u_2\log(u_1u_2) + (1-u_1)u_2\log((1-u_1)u_2) \\ & + u_1(1-u_2)\log(u_1(1-u_2)) + (1-u_1)(1-u_2)\log((1-u_1)(1-u_2)) \end{aligned}$$

and U_1, U_2 are i.i.d. with a uniform distribution on $(0,1)$. Here $\Sigma_1 = 1, \mathbb{E}\Sigma_2 = 4/9, C = h(U_1,U_2)$ and

$$T_1 = \max\{U_1U_2, U_1(1-U_2), (1-U_1)U_2, (1-U_1)(1-U_2)\}.$$

Since $\mathbb{P}[U_1U_2 \ge 1-\delta] = \frac{\delta^2}{2(1-\delta)}$, we have $\mathbb{P}[1-\delta \le T_1 \le 1] = \frac{2\delta^2}{1-\delta}$ for $0 < \delta < \frac{1}{2}$. Furthermore

$$\begin{aligned} \|h(U_1,U_2)^+\|_\infty &= \sup_{(u_1,u_2)\in(0,1)^2} h(u_1,u_2) = \lim_{(u_1,u_2)\uparrow(1,1)} h(u_1,u_2) = 1, \\ \|h(U_1,U_2)^-\|_\infty &= -\inf_{(u_1,u_2)\in(0,1)^2} h(u_1,u_2) = -h(1/2,1/2) = 2\log 2 - 1. \end{aligned}$$

For any $c \in (0,1)$ pick $\eta_c \in (1/2, 1]$ such that $h(\eta_c, \eta_c) = c$ and notice that for $\delta < 1 - \eta_c$

$$\begin{aligned}\delta^2 &= \mathbb{P}[U_1 > 1-\delta,\ U_2 > 1-\delta] = \mathbb{P}[U_1 > 1-\delta,\ U_2 > 1-\delta, C > c] \\ &\le \mathbb{P}[T_1 > 1-\delta,\ C > c].\end{aligned}$$

Having also the upper bound $\mathbb{P}[T_1 > 1-\delta,\ C > c] \le \mathbb{P}[1-\delta \le T_1 \le 1] = \frac{2\delta^2}{1-\delta}$, we arrive at the conclusion that

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[X^{(qt)} > x]}{x \log x} = -2 \quad \text{and} \tag{26}$$

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}[X^{(qt)} < -x]}{x \log x} \le -\frac{2}{2\log 2 - 1}. \tag{27}$$

Our next example is to demonstrate that, assuming nonnegative weights $T_k (\beta = 0)$, information about Σ_∞, i.e. the number of weights equal to 1, does not suffice to determine $\mathbb{D}_\Psi$. In some cases we rather need to know the behavior of their laws in small neighborhoods of 1.

Example 5.4. Pick any $\alpha < 2$ and let A be a random variable with a $\beta(\alpha, 1)$ distribution and thus density $\alpha t^{\alpha-1}\mathbf{1}_{(0,1)}(t)$. For any integer $n \ge 2$ satisfying $\alpha < \frac{2}{n-1}$, let further $N \equiv n, T_1 = \cdots = T_n = A$ and C be any random variable with mean 0 and independent of A. Then (4) reads

$$X \stackrel{d}{=} A \sum_{k=1}^{n} X_k + C$$

with associated functional equation (9) of the special form

$$\Psi(\theta) = \varphi(\theta) \int_0^1 \Psi(t\theta)^n \alpha t^{\alpha-1}\, dt$$

which in fact allows us to compute Ψ explicitly. By taking derivatives with respect to θ, we obtain

$$\begin{aligned}\Psi'(\theta) &= \varphi'(\theta) \int_0^1 \Psi(t\theta)^n \alpha t^{\alpha-1}\, dt + \varphi(\theta) \int_0^1 nt\ \Psi'(t\theta)\ \Psi(t\theta)^{n-1} \alpha t^{\alpha-1}\, dt \\ &= \frac{\varphi'(\theta)}{\varphi(\theta)}\ \Psi(\theta) + \frac{\varphi(\theta)}{\theta} \int_0^1 \frac{d}{dt}\left[\Psi(t\theta)^n\right] \alpha t^{\alpha}\, dt \\ &= \frac{\varphi'(\theta)}{\varphi(\theta)}\ \Psi(\theta) + \frac{\alpha\, \varphi(\theta)\, \Psi(\theta)^n}{\theta} - \frac{\varphi(\theta)}{\theta} \int_0^1 \Psi(t\theta)^n \alpha^2 t^{\alpha-1}\, dt \\ &= \frac{\varphi'(\theta)}{\varphi(\theta)}\ \Psi(\theta) + \frac{\alpha\, \varphi(\theta)\, \Psi(\theta)^n}{\theta} - \frac{\alpha\ \Psi(\theta)}{\theta}\end{aligned}$$

and therefore

$$\Psi'(\theta) = \frac{\alpha\, \varphi(\theta)}{\theta}\, \Psi(\theta)^n + \left(\frac{\varphi'(\theta)}{\varphi(\theta)} - \frac{\alpha}{\theta}\right) \Psi(\theta).$$

This is a Bernoulli differential equation and can be solved explicitly. Defining $x(\theta) := \Psi(\theta)^{1-n}$, this function satisfies

$$0 = x'(\theta) + (n-1)\frac{\alpha\varphi(\theta)}{\theta} + (n-1)\left(\frac{\varphi'(\theta)}{\varphi(\theta)} - \frac{\alpha}{\theta}\right) x(\theta)$$

from which we infer

$$0 = \frac{d}{d\theta}\left[\frac{\varphi(\theta)^{n-1}}{\theta^{\alpha(n-1)}}x(\theta)\right] + (n-1)\frac{\alpha\varphi(\theta)^n}{\theta^{\alpha(n-1)+1}}$$

and thereupon, for any pair (θ_0, θ) with $\theta_0 < \theta$,

$$\begin{aligned}\frac{\varphi(\theta)^{n-1}}{\theta^{\alpha(n-1)}}x(\theta) &= \frac{\varphi(\theta_0)^{n-1}}{\theta_0^{\alpha(n-1)}}x(\theta_0) \;-\; (n-1)\int_{\theta_0}^{\theta}\frac{\varphi(s)^n}{s^{\alpha(n-1)+1}}\,ds \\ &= \frac{1}{\theta_0^{\alpha(n-1)}}(1+o(1)) \;-\; (n-1)\int_{\theta_0}^{\theta}\frac{\varphi(s)^n}{s^{\alpha(n-1)+1}}\,ds \\ &= \left(\frac{1}{\theta^{\alpha(n-1)}} + \int_{\theta_0}^{\theta}\frac{\alpha(n-1)}{s^{\alpha(n-1)+1}}\,ds\right)(1+o(1)) - (n-1)\int_{\theta_0}^{\theta}\frac{\varphi(s)^n}{s^{\alpha(n-1)+1}}\,ds \\ &= \frac{1}{\theta^{\alpha(n-1)}} - \alpha(n-1)\int_{\theta_0}^{\theta}\frac{\varphi(s)^n - 1}{s^{\alpha(n-1)+1}}\,ds + o(1)\end{aligned}$$

where the $o(1)$ term is for $\theta_0 \to 0$ and fixed θ. Finally, by solving for $\Psi(\theta) = x(\theta)^{-1/(n-1)}$ and passing to the limit $\theta_0 \to 0$, we find

$$\Psi(\theta) = \frac{\varphi(\theta)}{\left(1 - \int_0^{\theta}(\varphi(s)^n - 1)\left(\frac{\theta}{s}\right)^{\alpha(n-1)+1}\alpha(n-1)\,ds\right)^{1/(n-1)}}.$$

With this explicit formula for Ψ, we see that $\mathbb{D}_\Psi$ is given by

$$\mathbb{D}_\Psi = \mathbb{D}_\varphi \cap \left\{\theta : \int_0^{\theta}(\varphi(s)^n - 1)\left(\frac{\theta}{s}\right)^{\alpha(n-1)+1}\alpha(n-1)\,ds < 1\right\} \tag{28}$$

and thus depends on $\mathbb{D}_\varphi$, the branching index n and, most notably, the parameter α which characterizes the tails of the T_k via

$$\mathbb{P}[T_k > t] = 1 - t^\alpha \quad \text{for } t \in (0, 1]. \tag{29}$$

As for $s < \theta$, the function

$$\alpha \;\mapsto\; \alpha\left(\frac{\theta}{s}\right)^{\alpha(n-1)+1}$$

is increasing, the set $\mathbb{D}_\Psi$ in (28) gets smaller, while the probabilities in (29) get bigger with increasing α.

Our last example shows that the cases (a3) and (a4) in Theorem 3.2 can actually occur.

Example 5.5. Let $N = 2$ and (C, T_1, T_2) take values

$$\left(-1, \frac{5}{4}, \frac{1}{4}\right) \quad \text{and} \quad \left(1, \frac{1}{4}, \frac{1}{4}\right)$$

with probability $\frac{1}{2}$ each. Then $\varphi(\theta) = \cosh\theta \le e^{|\theta|}$ for all $\theta \in \mathbb{R}$, $\|\Sigma_2\|_1 = \frac{7}{8} < 1 = \|\Sigma_1\|_1$, and

$$C + 2(\Sigma_1 - 1) = 0.$$

Obviously in the situation of case (3) in Theorem 3.2 with $w^* = 2$ if $\mathbb{D}_\Psi \neq \{0\}$, we claim that $\Psi_n(\theta) \le e^{2\theta}$ for all $\theta \ge 0$ and $n \in \mathbb{N}_0$ which, by Lemma 2.1, entails the same for $\Psi(\theta)$. For an induction over n, note that the claim holds for $\varphi = \Psi_0$. Assuming validity for Ψ_{n-1}, the backward equation (7) provides us with

$$\Psi_n(\theta) = \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Psi_{n-1}(\theta T_k)\right] \le e^{2\theta}\,\mathbb{E}e^{\theta(C+2(\Sigma_1-1))},$$

and since $C + 2(\Sigma_1 - 1) = 0$ the claim is proved.

Fixing any $p \in [1, 2]$, we can modify the previous example in such a way that (8) holds for this p while $\|C^-\|_\alpha = \infty$ for any $\alpha > p$. Namely, assume now that $N = 2$,

$$\mathbb{P}\left((C, T_1, T_2) = \left(1, \frac{1}{4}, \frac{1}{4}\right)\right) = \frac{2}{3},$$
$$\mathbb{P}\left((T_1, T_2) = \left(\frac{5}{4}, \frac{1}{4}\right)\right) = \frac{1}{3},$$
$$\mathbb{P}\left(C \in \cdot \,\middle|\, (T_1, T_2) = \left(\frac{5}{4}, \frac{1}{4}\right)\right) = \mathbb{P}(C' \in \cdot),$$

where $C' \in L^p$ takes values in $(-\infty, -1]$, has mean -2 and infinite absolute α-moments for $\alpha > p$. Then one can readily verify that $\mathbb{E}C = 0, \varphi(\theta) \le e^{\theta}$ for all $\theta \in \mathbb{R}_\geqslant$ (as $C \le 1$), $\|\Sigma_\alpha\|_1 < 1$ for all $\alpha \in [1, 2]$, and

$$C + 2(\Sigma_1 - 1) \le 0.$$

Therefore the above inductive argument still works to give $\Psi(\theta) \le e^{2\theta}$ for all $\theta \in \mathbb{R}_\geqslant$.

6. Proofs

Let us begin with a rather simple but useful technical lemma.

Lemma 6.1. *Suppose (8) for some $p \in [1, 2]$ and (10). Let $I \subset \mathbb{R}$ be an interval containing 0. Then $I \subset \mathbb{D}_\Psi$ iff there exists a function $\Phi : I \to [1, \infty)$, called supersolution of (9) on I,[1] such that $\Phi(0) = 1$ and*

$$\mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\right] \le \Phi(\theta) \quad \text{for all } \theta \in I. \tag{30}$$

In this case $\Psi \le \Phi$ on I.

Proof. Suppose there is a supersolution Φ and let $W_{-1} := 0$. Then we have

$$\Psi_{-1}(\theta) := \mathbb{E}e^{\theta W_{-1}} = 1 \le \Phi(\theta)$$

for all $\theta \in I$. Now use induction over n. Assuming $\Psi_{n-1} \le \Phi$ on I, (30) and the backward Eq. (7), we obtain

$$\Psi_n(\theta) = \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Psi_{n-1}(T_k\theta)\right] \le \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\right] \le \Phi(\theta)$$

[1] And in fact a superharmonic function for the smoothing transform $\mathcal{S}$ when viewed as an operator on the halfspace of functions $f : I \to \mathbb{R}_\geqslant$ and defined by $\mathcal{S}f(\theta) := \mathbb{E}[e^{\theta C}\prod_{k=1}^{N} f(T_k\theta)]$ for $\theta \in I$.

and therefore, by Lemma 2.1,

$$\Psi(\theta) = \lim_{n\to\infty} \Psi_n(\theta) \le \Phi(\theta) \ < \ \infty$$

for all $\theta \in I$.

Conversely, if $\mathbb{D}_\Psi \supset I$, then Ψ itself is a supersolution. $\square$

In order to prove Theorem 3.2, we will actually prove the following slightly more general result.

Theorem 6.2. *Suppose that (8) holds for some $p \in [1, 2]$.*

(a) *If $\mathbb{D}_\Psi \ne \{0\}$, then one of the following four cases occurs:*
  (a1) *(10) holds, $\beta = 0$, and $\mathbb{D}_\varphi \ne \{0\}$.*
  (a2) *(10) holds, $\beta > 0$, and $\mathbb{D}_\varphi \supset (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.*
  (a3) *$\mathbb{P}[T_1 > 1] > 0$, $\beta = 0$, and $\mathbb{P}\left[w^*\Sigma_1 + C \le w^*\right] = 1$ for some $w^* \ge 0$.*
  (a4) *$\mathbb{P}[T_1 > 1] > 0$, $\beta = 0$, and $\mathbb{P}\left[w_*\Sigma_1 + C \ge w_*\right] = 1$ for some $w_* \le 0$.*
  Moreover, $\mathbb{D}_\Psi \supset (-\varepsilon, \varepsilon) \cap \mathbb{D}_\varphi$ for some $\varepsilon > 0$ in the first two cases, while $\|C^+\|_\infty \le \|W^+\|_\infty < \infty, \mathbb{D}_\Psi = \mathbb{R}_\geqslant$ must hold in case (a3) *and $\|C^-\|_\infty \le \|W^-\|_\infty < \infty, \mathbb{D}_\Psi = \mathbb{R}_\leqslant$ must hold in case* (a4).

(b) *Conversely,* (a3) *and* (a4) *imply $\mathbb{D}_\Psi \ne \{0\}$, and this is also true for* (a1) *and* (a2) *under the additional assumption (12).*

Proof. (a) Suppose $\mathbb{D}_\Psi \ne \{0\}$ and pick any $\theta \in \mathbb{D}_\Psi \setminus \{0\}$. By Lemma 2.1, $\{0\} \ne \mathbb{D}_\Psi \subset \mathbb{D}_\varphi$, thus (a1) is valid if also (10) and $\beta = 0$ are assumed.

Next consider the case when (10) holds, but $\beta > 0$, that is $\mathbb{P}[T_N < 0] > 0$. Then $\mathbb{P}[T_N < -\delta] > 0$ for some $\delta \in (0, 1)$ and hence, by (9),

$$\infty > \Psi(\theta) = \mathbb{E}\left[e^{\theta C} \prod_{k=1}^{N} \Psi(T_k\theta)\right] \ge \mathbb{E}\left[e^{\theta C} \mathbf{1}_{\{T_N < -\delta\}}\right] \Psi(-\delta\theta), \tag{31}$$

giving $-\delta\theta \in \mathbb{D}_\Psi$ and thereupon $[-\delta\theta, \delta\theta] \subset \mathbb{D}_\Psi \subset \mathbb{D}_\varphi$, for $\mathbb{D}_\Psi$ is convex. In other words, the conditions of (a2) are valid.

Finally assume that (10) fails to hold, thus

$$|T_1| \vee |T_N| = \max_{1\le k\le N} |T_k| > 1 \quad \text{with positive probability.}$$

This is the most difficult situation and requires some work. Further assuming $\theta > 0$, we will show now that the conditions of (a3) are valid. By an analogous argument, those of (a4) follow if $\theta < 0$.

Claim 1. $\mathbb{P}[T_N \ge 0] = 1$ *and thus* $\beta = 0$.

Assuming the contrary, another use of (31) yields $[-\delta\theta, \delta\theta] \subset \mathbb{D}_\Psi$, thus $\mathbb{E}e^{\delta\theta|W|} < \infty$, which contradicts Proposition 3.1. Consequently, $T_N \ge 0$ a.s. which in turn implies $\beta = 0$ and then $T_1 = \max_{1\le k\le N} T_k > 1$ with positive probability.

Claim 2. $\mathbb{D}_\Psi = \mathbb{R}_\geqslant$.

Choose $\varepsilon > 0$ such that $\gamma := \mathbb{P}[T_1 > 1 + \varepsilon] > 0$. By another use of (9), we infer that

$$\infty > \Psi(\theta) = \mathbb{E}\left[e^{\theta C} \prod_{k=1}^{N} \Psi(T_k\theta)\right] \ge \mathbb{E}\left[e^{\theta C} \mathbf{1}_{\{T_1 > 1+\varepsilon\}}\right] \Psi((1+\varepsilon)\theta) \tag{32}$$

and thus $(1+\varepsilon)\theta \in \mathbb{D}_\Psi$. Iterating this argument, we obtain $\mathbb{R}_\geqslant \subset \mathbb{D}_\Psi$. By another appeal to Proposition 3.1, we must have $\mathbb{D}_\Psi = \mathbb{R}_\geqslant$.

Claim 3. $\mathbb{P}[C \le 0|T_1 > 1] = 1$.

If $\mathbb{P}[C > 0|T_1 > 1] > 0$, then (32) remains valid with $\mathbb{E}[e^{\theta C}\mathbf{1}_{\{T_1>1\}}]\Psi(\theta) > 0$ on the right-hand side and we arrive at the impossible conclusion that

$$1 \ge \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1>1\}}\right] \ge \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{C\ge 0, T_1>1\}}\right] \xrightarrow{\theta\to\infty} \infty$$

(having used $\mathbb{D}_\Psi = \mathbb{R}_\geqslant$).

Claim 4. *W is a.s. bounded from above, i.e.* $\|W^+\|_\infty < \infty$.

Assuming the contrary, i.e. $\|W^+\|_\infty = \infty$, it is a well-known fact that $\log \Psi$ is an increasing strictly convex function on its domain $\mathbb{D}_\Psi = \mathbb{R}_\geqslant$, whence its derivative $\Psi'(\theta)/\Psi(\theta)$ increases to ∞ as $\theta \to \infty$. As a consequence,

$$\frac{1}{\varepsilon\theta}\log\left(\frac{\Psi((1+\varepsilon)\theta)}{\Psi(\theta)}\right) = \frac{\log \Psi((1+\varepsilon)\theta) - \log \Psi(\theta)}{\varepsilon\theta} \ge \frac{\Psi'(\theta)}{\Psi(\theta)} \xrightarrow{\theta\to\infty} \infty$$

for any fixed $\varepsilon > 0$. In other words, $\Psi((1+\varepsilon)\theta)/\Psi(\theta) = e^{\theta h(\theta)}$ for some function h satisfying $\lim_{\theta\to\infty} h(\theta) = \infty$. Now let ε and γ be as under Claim 2 and use $\nu := \mathbb{E}[C|T_1 > 1+\varepsilon] \le 0$ by Claim 3 in combination with Jensen's inequality to infer

$$\mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1>1+\varepsilon\}}\right] \ge \gamma\,\mathbb{E}\left[e^{\theta C}|T_1 > 1+\varepsilon\right] \ge \gamma\, e^{\theta\nu}$$

for all $\theta \ge 0$. Returning to (32) and using the previous facts, we arrive at the contradiction

$$1 \ge \frac{\Psi((1+\varepsilon)\theta)}{\Psi(\theta)}\,\mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1>1+\varepsilon\}}\right] \ge e^{\theta(\nu+h(\theta))} \xrightarrow{\theta\to\infty} \infty.$$

Claim 5. $\mathbb{P}[w^*\Sigma_1 + C \le w^*] = \mathbb{P}[C \le w^*] = 1$ *for* $w^* := \|W^+\|_\infty$.

Since $(C, T_1, T_2, \ldots)$ and $(W, W_1, W_2, \ldots)$ are independent and all T_k are nonnegative ($\beta = 0$), the SFPE (4) provides us with

$$w^* = \operatorname{ess\,sup}\left(\sum_{k=1}^{N} T_k W_k + C\right) = \operatorname{ess\,sup}\left(w^*\Sigma_1 + C\right)$$

which in turn implies

$$1 = \mathbb{P}[w^*\Sigma_1 + C \le w^*] \le \mathbb{P}[C \le w^*]$$

as claimed.

(b) It remains to show that each of the cases (a1)–(a4), the first two under the additional assumption (12), implies $\mathbb{D}_\Psi \ne \{0\}$ and that even $\mathbb{D}_\Psi \supset (-\varepsilon, \varepsilon) \cap \mathbb{D}_\varphi$ for some $\varepsilon > 0$ holds true under (a1) and (a2).

If (a3) holds, then $w^*\Sigma_1 + C \le w^*$ a.s. For some $w^* \ge 0$ entails $C \le w^*$ a.s. because all T_k are nonnegative. By using the backward equation (7) inductively, we then obtain $W_n \le w^*$ a.s. and thereupon $W \le w^*$ a.s. which in turn implies $\mathbb{D}_\Psi \supset \mathbb{R}_\geqslant$. A similar argument shows $\mathbb{D}_\Psi \supset \mathbb{R}_\leqslant$ if (a4) is valid.

Left with the cases (a1) and (a2), which are treated together, we first note that in case (a1) we may assume w.l.o.g. that $\mathbb{D}_\varphi \cap \mathbb{R}_> \neq \emptyset$, for otherwise we may switch to the smoothing transform based on $(-C, T_1, T_2, \ldots)$ and with canonical fixed point $-W$. We further note that (8) for some $p \in [1, 2]$ combined with (10) entails $\|\Sigma_2\|_1 < 1$. Recall that $\sigma_W^2 = \mathbb{V}\mathrm{ar}\, W = \mathbb{E}C^2/(1 - \mathbb{E}\Sigma_2)$ is finite. For $\theta \in \mathbb{R}$, consider now the random function

$$G(\theta) := e^{\theta C + b\theta^2(\Sigma_2 - 1)},$$

where the constant $b > \sigma_W^2/2$ is chosen in such a way that

$$\mathbb{E}C^2 + b(\mathbb{E}\Sigma_2 - 1) \;<\; 0.$$

The first three derivatives of G with respect to θ are given by

$$\begin{aligned}
G'(\theta) &= \big(C + 2\theta b(\Sigma_2 - 1)\big)G(\theta),\\
G''(\theta) &= \Big(\big(C + 2\theta b(\Sigma_2 - 1)\big)^2 C + 2b(\Sigma_2 - 1)\Big)\, G(\theta),\\
G'''(\theta) &= \big(4b(\Sigma_2 - 1) + 1\big)\big(C + 2b\theta(\Sigma_2 - 1)\big)G(\theta),
\end{aligned}$$

so that

$$G(0) = 1, \quad G'(0) = C \quad \text{and} \quad G''(0) = C^2 + 2b(\Sigma_2 - 1).$$

By (12), we can fix $\theta_0 \in \mathbb{D}_\varphi \cap \mathbb{R}_>$ sufficiently small such that, with the help of Hölder's inequality,

$$\mathbb{E}G(2\theta_0) \le \varphi(4\theta_0)^{1/2} \left(\mathbb{E}e^{8b\theta_0^2(\Sigma_2 - 1)}\right)^{1/2} \;<\; \infty.$$

For $\theta \in (-\theta_0, \theta_0)$, we then obtain

$$G'''(\theta) \le \big|4b(\Sigma_2 - 1) + 1\big|\big(|C| + 2b\theta_0|\Sigma_2 - 1|\big)e^{\theta_0 C^+ + b\theta_0^2(\Sigma_2 - 1)} \;=:\; \overline{G}(\theta_0),$$

and $\mathbb{E}\overline{G}(\theta_0) < \infty$. A third-order Taylor expansion of $\mathbb{E}G(\theta)$ about 0 provides us with

$$\begin{aligned}
\mathbb{E}G(\theta) &\le G(0) + \theta\, \mathbb{E}G'(0) + \frac{\theta^2}{2}\, \mathbb{E}G''(0) + \frac{|\theta|^3}{6}\, \mathbb{E}\overline{G}(\theta_0)\\
&= 1 + \frac{\theta^2}{2}\, \Big(\mathbb{E}C^2 + 2b(\mathbb{E}\Sigma_2 - 1)\Big) + \frac{|\theta|^3}{6}\, \mathbb{E}\overline{G}(\theta_0)
\end{aligned}$$

for all sufficiently small $\theta \in \mathbb{D}_\varphi$. By the choice of b, we can now fix $\delta > 0$ such that for any $\theta \in I := (-\delta, \delta) \cap \mathbb{D}_\varphi$, we have

$$\mathbb{E}G(\theta) \le 1.$$

But this implies that the function $\Phi : I \to [1, \infty]$, $\Phi(\theta) := e^{b\theta^2}$ satisfies condition (30) of Lemma 6.1 and thus leads to the conclusion that $I \subset \mathbb{D}_\Psi$ as asserted. □

Notice that form the last proof, $\Psi(\theta) \le e^{b\theta^2}$ for sufficiently small $\theta \in \mathbb{D}_\Psi$. Since this inequality in also valid for θ bounded away from 0 by increasing b, we may infer that for any bounded $I \subseteq \mathbb{D}_\Psi$ one can always pick B_I large enough such that $\Psi(\theta) \le e^{B_I\theta^2}$ for $\theta \in I$.

For the proofs of Theorems 3.4 and 3.5, the main work is provided by two subsequent lemmata so as to keep the presentation as transparent as possible.

Lemma 6.3. *Suppose $\beta = 1$ and the assumptions of Theorem 3.5 be satisfied. Then, for any $\theta \in \mathbb{R}$, $-\theta, \theta \in \mathbb{D}_\Psi$ iff*

$$(1-a(\theta))(1-a(-\theta)) > b(\theta)b(-\theta) \quad \textit{and} \quad [-\theta,\theta] \subset \mathbb{D}_\varphi.$$

Proof. Put

$$\widehat{\mathbb{D}}_\Psi := \{\theta \in \mathbb{R} : -\theta, \theta \in \mathbb{D}_\Psi\}$$

and

$$\mathbb{D} := \{\theta \in \mathbb{R} : (1-a(\theta))(1-a(-\theta)) > b(\theta)b(-\theta) \text{ and } -\theta, \theta \in \mathbb{D}_\varphi\},$$

so that $\widehat{\mathbb{D}}_\Psi = \mathbb{D}$ must be verified. Note that both sets are symmetric about 0 by definition. By Theorem 3.2, $\mathbb{D} \neq \{0\}$ and thus $0 \in int(\mathbb{D}_\varphi)$ entails $\mathbb{D}_\Psi \supset (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ (Case (a2) there).

For the inclusion $\widehat{\mathbb{D}}_\Psi \subset \mathbb{D}$, let $\theta \in \widehat{\mathbb{D}}_\Psi$. Then $[-\theta,\theta] \subset \mathbb{D}_\varphi$ by an appeal to Lemma 2.1. Using the functional equation (9), we find

$$\begin{aligned}\Psi(\theta) &\geq \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1=1\}}\right]\Psi(\theta) + \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_N=-1\}}\right]\Psi(-\theta) + \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{|T_1|,|T_N|<1\}}\right]\\ &> \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1=1\}}\right]\Psi(\theta) + \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_N=-1\}}\right]\Psi(-\theta)\end{aligned}$$

as well as

$$\Psi(-\theta) > \mathbb{E}\left[e^{-\theta C}\mathbf{1}_{\{T_1=1\}}\right]\Psi(-\theta) + \mathbb{E}\left[e^{-\theta C}\mathbf{1}_{\{T_N=-1\}}\right]\Psi(\theta),$$

and these inequalities may be rewritten as

$$\Psi(\theta)(1-a(\theta)) > b(\theta)\,\Psi(-\theta)$$

and

$$\Psi(-\theta)(1-a(-\theta)) > b(-\theta)\,\Psi(\theta),$$

respectively. Since all appearing quantities are positive and finite, multiplication yields

$$\Psi(\theta)\,\Psi(-\theta)(1-a(\theta))(1-a(-\theta)) > \Psi(-\theta)\,\Psi(\theta)b(\theta)b(-\theta)$$

and thus $\theta \in \mathbb{D}$.

Having just shown $\widehat{\mathbb{D}}_\Psi \subset \mathbb{D}$, suppose now that this inclusion is strict, i.e. $\mathbb{D} \setminus \widehat{\mathbb{D}}_\Psi \neq \emptyset$, in particular $\mathbb{D} \neq \{0\}$ and thus

$$\mathbb{D}_\Psi \supset \widehat{\mathbb{D}}_\Psi \supset (-\varepsilon, \varepsilon) \quad \text{for some } \varepsilon > 0. \tag{33}$$

By our assumptions, there exists $\delta_1 > 0$ such that

$$\left\|\max_{2\leq k\leq N} T_k^+\right\|_\infty \vee \left\|\max_{1\leq k\leq N-1} T_k^-\right\|_\infty < 1-\delta_1.$$

Define $\theta_0 := \inf \mathbb{R}_> \cap (\mathbb{D} \setminus \widehat{\mathbb{D}}_\Psi)$, which is positive by (33). Moreover, it follows that $[-(1-\delta_1)\theta_0, (1-\delta_1)\theta_0] \subset \widehat{\mathbb{D}}_\Psi \cap int(\mathbb{D}_\Psi)$ and

$$\frac{1-a(\theta_0)}{b(\theta_0)} > \eta > \frac{b(-\theta_0)}{1-a(-\theta_0)}$$

for some $\eta > 0$. For $\delta > 0$, consider

$$a_\delta(\theta_0) := \mathbb{E}\left[e^{\theta_0 C}\prod_{k=2}^{N}\Psi(T_k\theta_0)\mathbf{1}_{\{T_1\in(1-\delta,1]\}}\right].$$

Since $(1-\delta_1)\theta_0 \in int(\mathbb{D}_\Psi)$, we have that $\log\Psi(\theta) \le \kappa\theta^2$ for all $\theta \in [0,(1-\delta_1)\theta_0]$ and some $\kappa = \kappa(\theta_0) > 0$. As a consequence,

$$e^{\theta_0 C}\prod_{k=2}^{N}\Psi(T_k\theta_0)\mathbf{1}_{\{T_1\in(1-\delta,1]\}} \le e^{\theta_0 C+\kappa\theta_0^2\Sigma_2} \quad \text{a.s.}$$

and thereby (using $\|\Sigma_2\|_\infty < \infty$ and $\mathbb{D}_\Psi \subset \mathbb{D}_\varphi$)

$$\mathbb{E}\left[e^{\theta_0 C+\kappa\theta_0^2\Sigma_2}\right] \le \varphi(\theta_0)\, e^{\kappa\theta_0^2\|\Sigma_2\|_\infty} < \infty.$$

With the help of the dominated convergence theorem, we now infer that

$$a_\delta(\theta_0) \stackrel{\delta\to 0}{\longrightarrow} a(\theta_0) < 1,$$

and by a similar argument also

$$b_\delta(\theta_0) := \mathbb{E}\left[e^{\theta_0 C}\prod_{k=1}^{N-1}\Psi(T_k\theta_0)\mathbf{1}_{\{T_N\in[-1,-1+\delta)\}}\right] \stackrel{\delta\to 0}{\longrightarrow} b(\theta_0)$$

and the corresponding assertions with $-\theta_0$ instead of θ_0. Therefore, we can pick $\delta \in (0,\delta_1)$ such that

$$\frac{1-a_\delta(\theta_0)}{b_\delta(\theta_0)} > \eta > \frac{b_\delta(-\theta_0)}{1-a_\delta(-\theta_0)},$$

and then by continuity further $\theta_* \in \widehat{\mathbb{D}}_\Psi$ and $\theta^* \in \mathbb{D}\setminus\widehat{\mathbb{D}}_\Psi$ such that $(1-\delta)\theta^* < \theta_* < \theta_0 \le \theta^*$ and

$$\frac{1-a_\delta(\theta)}{b_\delta(\theta)} > \eta > \frac{b_\delta(-\theta)}{1-a_\delta(-\theta)} \quad \text{for all } \theta \in [\theta_*,\theta^*].$$

Let δ also be small enough to guarantee (16) and (17) of Theorem 3.5.

Consider the function $\Phi : [-\theta^*,\theta^*] \to [1,\infty)$, defined by

$$\Phi(\theta) := \begin{cases} \xi\eta, & \text{if } \theta \in [-\theta^*,-\theta_*), \\ \Psi(\theta), & \text{if } \theta \in [-\theta_*,\theta_*], \\ \xi, & \text{if } \theta \in (\theta_*,\theta^*] \end{cases} \tag{34}$$

for some $\xi \ge \Psi(\theta_*) \vee \eta^{-1}\Psi(-\theta^*)$. As will be shown next, ξ can be chosen in such a way that Φ satisfies (30). To this end we point out first that $\Phi = \Psi$ on $[-\theta_*,\theta_*]$ in combination with (10) ensures

$$\mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\right] = \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Psi(T_k\theta)\right] = \Psi(\theta) = \Phi(\theta)$$

for all $\theta \in [-\theta_*,\theta_*]$ whence we need to verify (30) for $|\theta| \in (\theta_*,\theta^*]$. Put

$$M := \left\{\max_{1\le k\le N}|T_k| \le 1-\delta\right\} \quad \text{and} \quad c_\delta(\theta) := \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\mathbf{1}_M\right]$$

and note that by (16) and (17), if $T_1 > 1-\delta$, then $|T_k| < 1-\delta$ a.s. For $k \neq 1$, thus

$$T_1 > 1-\delta \quad \Rightarrow \quad T_k\theta \in [-\theta_*, \theta_*] \quad \text{a.s. for } |\theta| \le |\theta^*| \text{ and } 2 \le k \le N,$$

where $(1-\delta)\theta^* < \theta_*$ should be recalled. By an analogous argument,

$$T_N < -1+\delta \quad \Rightarrow \quad T_k\theta \in [-\theta_*, \theta_*] \quad \text{a.s. for } |\theta| \le |\theta^*| \text{ and } 1 \le k \le N-1.$$

With these observations, we arrive at the inequality

$$\begin{aligned}
\mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\right] &\le \Phi(\theta)\,\mathbb{E}\left[e^{\theta C}\prod_{k=2}^{N}\Phi(T_k\theta)\mathbf{1}_{\{T_1>1-\delta\}}\right] \\
&\quad + \Phi(-\theta)\,\mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N-1}\Phi(T_k\theta)\mathbf{1}_{\{T_N<-1+\delta\}}\right] \\
&\quad + \mathbb{E}\left[e^{\theta C}\prod_{k=1}^{N}\Phi(T_k\theta)\mathbf{1}_M\right] \\
&= \Phi(\theta)a_\delta(\theta) + \Phi(-\theta)b_\delta(\theta) + c_\delta(\theta),
\end{aligned}$$

valid for $\theta \in [-\theta^*, \theta^*]$. So we must verify that ξ can be chosen in such a way that, for $|\theta| \in (\theta_*, \theta^*]$,

$$\Phi(\theta)a_\delta(\theta) + \Phi(-\theta)b_\delta(\theta) + c_\delta(\theta) \le \Phi(\theta). \tag{35}$$

or, equivalently,

$$\xi a_\delta(\theta) + \xi\eta b_\delta(\theta) + c_\delta(\theta) \le \xi \quad \text{and} \tag{36}$$

$$\xi\eta a_\delta(-\theta) + \xi b_\delta(-\theta) + c_\delta(-\theta) \le \xi\eta \tag{37}$$

for $\theta \in (\theta_*, \theta^*]$. For (36), this obviously requires

$$\xi \ge \sup_{\theta\in[-\theta_*,\theta^*]} \frac{c_\delta(\theta)}{(1-a_\delta(\theta)) - \eta b_\delta(\theta)},$$

while (37) requires

$$\xi \ge \sup_{\theta\in[-\theta_*,\theta^*]} \frac{c_\delta(-\theta)}{\eta(1-a_\delta(-\theta)) - b_\delta(-\theta)}.$$

Since both suprema are positive and finite, we can choose ξ to be smallest number in $[\ge \Psi(\theta_*) \vee \eta^{-1}\Psi(-\theta^*), \infty)$ satisfying both inequalities. Then Φ defined by (34) satisfies (30) of Lemma 6.1, whence this lemma implies $[-\theta^*, \theta^*] \subset \mathbb{D}_\Psi$, i.e. $\theta^* \in \widehat{\mathbb{D}}_\Psi$, which is a contradiction. □

The proof of the next lemma differs from the previous one only in some technical aspects and we therefore supply details only where necessary.

Lemma 6.4. *Suppose $\beta < 1$ and the assumptions of Theorem 3.4 ($\beta = 0$) or Theorem 3.5 ($\beta > 0$) be satisfied. Then, for any $\theta \in \mathbb{R}$, $-\beta\theta, \theta \in \mathbb{D}_\Psi$ iff*

$$a(\theta) \vee a(-\beta\theta) < 1 \quad \textit{and} \quad -\beta\theta, \theta \in \mathbb{D}_\varphi.$$

Proof. Here we put

$$\widehat{\mathbb{D}}_\Psi := \{\theta \in \mathbb{R} : -\beta\theta, \theta \in \mathbb{D}_\Psi\}$$

and

$$\mathbb{D} := \{\theta \in \mathbb{R} : a(\theta) \vee a(-\beta\theta) < 1 \text{ and } -\beta\theta, \theta \in \mathbb{D}_\varphi\},$$

and must again verify $\widehat{\mathbb{D}}_\Psi = \mathbb{D}$.

For the proof of $\widehat{\mathbb{D}}_\Psi \subset \mathbb{D}$, pick any $\theta \in \widehat{\mathbb{D}}_\Psi$. Then (9) provides us with

$$\Psi(\theta) > \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_1=1\}}\right]\Psi(\theta) + \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_N=-\beta\}}\right]\Psi(-\beta\theta)$$

and

$$\Psi(-\beta\theta) > \mathbb{E}\left[e^{-\beta\theta C}\mathbf{1}_{\{T_1=1\}}\right]\Psi(-\beta\theta) + \mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_N=-\beta\}}\right]\Psi(\beta^2\theta),$$

which in turn lead to

$$\Psi(\theta)(1-a(\theta)) > 0 \quad \text{and} \quad \Psi(-\beta\theta)(1-a(-\beta\theta)) > 0,$$

respectively. This obviously proves the asserted inclusion.

Assuming this inclusion to be proper, thus $\mathbb{D} \setminus \widehat{\mathbb{D}}_\Psi \neq \emptyset$ and $\mathbb{D} \neq \{0\}$, we infer validity of (33) as in the previous lemma by an appeal to Theorem 3.2 (Cases (a1) or (a2)). Note also that, by our assumptions, there exists $\delta_1 \in (0, 1-\beta) \neq \emptyset$ such that

$$\left\|\max_{2\le k\le N} T_k^+\right\|_\infty < 1-\delta_1 \quad \text{and} \quad \left\|\max_{1\le k\le N-1} T_k^-\right\|_\infty \le (1-\delta_1)\beta.$$

Once again, $\theta_0 := \inf \mathbb{R}_> \cap (\mathbb{D} \setminus \widehat{\mathbb{D}}_\Psi)$ is positive by (33), and we have further that $[-\beta(1-\delta_1)\theta_0, (1-\delta_1)\theta_0] \subset \widehat{\mathbb{D}}_\Psi \cap int(\mathbb{D}_\Psi)$ and

$$1 - a(\theta_0) > \eta b(\theta_0)$$

for some $\eta > 0$. As argued in the previous proof,

$$a_\delta(\theta_0) = \mathbb{E}\left[e^{\theta_0 C}\prod_{k=2}^N \Psi(T_k\theta_0)\mathbf{1}_{\{T_1\in(1-\delta,1]\}}\right] \xrightarrow{\delta\to 0} a(\theta_0) < 1,$$

and

$$b_\delta(\theta_0) := \mathbb{E}\left[e^{\theta_0 C}\prod_{k=2}^N \Psi(T_k\theta_0)\mathbf{1}_{\{T_N\in[-\beta,-(1-\delta)\beta)\}}\right] \xrightarrow{\delta\to 0} b(\theta_0).$$

Therefore, we can pick $\delta \in (0, \delta_1)$ such that (notice the difference here to the previous proof)

$$1 - a_\delta(\theta_0) > \eta b_\delta(\theta_0) \quad \text{and} \quad a_\delta(-\beta\theta_0) < 1,$$

and then $\theta_* \in \widehat{\mathbb{D}}_\Psi$ and $\theta^* \in \mathbb{D} \setminus \widehat{\mathbb{D}}_\Psi$ such that $(1-\delta)\theta^* < \theta_* < \theta_0 \le \theta^*$ and

$$1 - a_\delta(\theta) > \eta b_\delta(\theta) \quad \text{and} \quad a_\delta(-\beta\theta) < 1 \quad \text{for all } \theta \in [\theta_*, \theta^*].$$

Again, let δ also be small enough to guarantee (16) and (17) of Theorem 3.5 if $\beta > 0$, and $\|\max_{2\le k\le N} T_k\|_\infty < 1-\delta$ if $\beta = 0$. Note that since $1-\delta < 1-\delta_1 < \beta < 1$, then $\beta^2\theta^* < \theta_*$.

Defining the function $\Phi : [-\theta^*, \theta^*] \to [1, \infty)$ by

$$\Phi(\theta) := \begin{cases} \xi\eta, & \text{if } \theta \in [-\beta\theta^*, -\beta\theta_*), \\ \Psi(\theta), & \text{if } \theta \in [-\beta\theta_*, \theta_*], \\ \xi, & \text{if } \theta \in (\theta_*, \theta^*] \end{cases}$$

for some $\xi \geq \Psi(\theta_*) \vee \eta^{-1}\Psi(-\beta\theta^*)$, the remaining proof follows almost the same lines as the previous one with lines (35)–(37) replaced by

$$\Phi(\theta)a_\delta(\theta) + \Phi(-\beta\theta)b_\delta(\theta) + c_\delta(\theta) \leq \Phi(\theta) \tag{38}$$

$$\xi a_\delta(\theta) + \eta\xi b_\delta(\theta) + c_\delta(\theta) \leq \xi \tag{39}$$

$$\eta\xi a_\delta(-\beta\theta) + \Psi(\theta_*)b_\delta(-\beta\theta) + c_\delta(-\beta\theta) \leq \eta\xi \tag{40}$$

respectively. We arrive at the same conclusion that, for suitable ξ, Φ satisfies (30) of Lemma 6.1, thus producing the contradiction $[-\beta\theta^*, \theta^*] \subset \mathbb{D}_\Psi$, i.e. $\theta^* \in \widehat{\mathbb{D}}_\Psi$. Further details are therefore omitted. $\square$

Proof of Theorem 3.4. This result now follows directly from Lemma 6.4 for the case $\beta = 0$. $\square$

Proof of Theorem 3.5. Here a separate discussion of the two cases $\mathbb{P}[T_N = -\beta] > 0$ and $\mathbb{P}[T_N = -\beta] = 0$ is necessary.

If the first alternative occurs, then $b(\theta) = \mathbb{E}[e^{\theta C}\mathbf{1}_{\{T_N=-\beta\}}] > 0$ for any $\theta \in \mathbb{D}_\Psi$. By another appeal to (9), we then infer

$$\infty > \Psi(\theta) \geq \Psi(-\beta\theta)\, b(\theta) > 0$$

for any $\theta \in \mathbb{D}_\Psi$ and thereby $-\beta\theta, \theta \in \mathbb{D}_\Psi$. The assertion now follows from Lemma 6.3 if $\beta = 1$, and from Lemma 6.4 if $\beta < 1$.

If $\mathbb{P}[T_N = -\beta] = 0$, then (9) provides us with

$$\infty > \Psi((1-\varepsilon)^{-1}\theta) \geq \Psi(-\beta\theta)\,\mathbb{E}\left[e^{\theta C}\mathbf{1}_{\{T_N \in (-\beta, -(1-\varepsilon)\beta]\}}\right] > 0$$

for any $\theta \in int(\mathbb{D}_\Psi)$ and some $\varepsilon \in (0, 1)$ such that $(1-\varepsilon)^{-1}\theta \in \mathbb{D}_\Psi$, thus giving $-\beta\theta \in \mathbb{D}_\Psi$. The assertion finally follows as before from Lemma 6.3 if $\beta = 1$, and from Lemma 6.4 if $\beta < 1$. $\square$

Proof of Theorem 4.1. The proof consists of two steps, establishing the upper bound and, assuming (20), the lower bound. Note that, under the given assumptions, $\mathbb{R}_\geqslant \subseteq \mathbb{D}_\varphi$, $\mathbb{P}[T_1 = 1] = 0$ and thus $a(\theta) = 0$ for all $\theta \in \mathbb{R}$. Consequently, by Theorem 3.4, $\mathbb{R}_\geqslant \subseteq \mathbb{D}_\Psi$.

UPPER BOUND. Consider the function $\Phi : \mathbb{R}_\geqslant \to [1, \infty)$, defined by

$$\Phi(\theta) := \begin{cases} \Psi(\theta), & \text{if } \theta \in [0, 1], \\ \exp(\xi\theta^q e^{b\theta}), & \text{if } \theta \in (1, \infty), \end{cases}$$

with $b := c^+/\gamma$, q such that $\|\Sigma_q\|_\infty \leq 1$, and $\xi > 0$. We claim that ξ can be chosen so large such that Φ is a supersolution of (9) on $\mathbb{R}_\geqslant$, i.e., satisfies (30) on this set. Since this is plain for

$\theta \le 1$, we must only consider $\theta > 1$. Put

$$\zeta := \sup_{\theta \in [0,1]} \frac{\log \Psi(\theta)}{\theta^2}.$$

As $\Phi(\theta) \le \exp(\zeta\theta^2 + \xi\theta^q e^{b\theta})$ for all $\theta \ge 0$, it suffices to verify

$$\mathbb{E}\left[\exp\left(\theta C + \zeta\theta^2 \Sigma_2 + \xi\theta^q \sum_{k=1}^{N} T_k^q e^{bT_k\theta} - \xi\theta^q e^{b\theta}\right)\right] \le 1 \quad \text{for } \theta > 1.$$

For any $t \in (0, 1]$, the expectation on the right-hand side is bounded by

$$\mathbb{E}\left[\mathbf{1}_{\{T_1 > t\}} \exp\left(\theta c^+ + \zeta \Sigma_2 + \xi\theta^q \sum_{k=2}^{N} T_k^q e^{bT_k\theta}\right)\right] \\ + \mathbb{E}\left[\mathbf{1}_{\{T_1 \le t\}} \exp\left(\theta c^+ + \zeta \Sigma_2 + \xi\theta^q \Sigma_q e^{bT_1\theta} - \xi\theta^q e^{b\theta}\right)\right]. \tag{41}$$

Pick $\rho \in (0, 1/2D)$ with $\gamma,\ D$ as in (19) and put

$$t = t(\theta, \rho) := 1 - \left(\frac{\rho}{D}\right)^{1/\gamma} e^{-\theta c^+/\gamma}.$$

Then, by using the assumptions of the theorem, the first term in (41) can be bounded by

$$D(1-t)^\gamma \exp\left(\theta c^+ + \zeta\|\Sigma_2\|_\infty + \xi\theta^q(1-t^q)\exp\left(b(1-t^q)^{1/q}\theta\right)\right) \\ = \rho\ \exp\left(\zeta\|\Sigma_2\|_\infty + \xi\theta^q(1-t^q)\exp\left(b(1-t^q)^{1/q}\theta\right)\right).$$

Since $1 - t^q \le e^{-\theta c^+ q/\gamma}$, we further see that

$$\zeta\|\Sigma_2\|_\infty + \xi\theta^q(1-t^q)\exp\left(b(1-t^q)^{1/q}\theta\right) \le c_1 < \infty \quad \text{for all } \theta > 1$$

and thus obtain, by choosing ρ sufficiently small,

$$\mathbb{E}\left[\mathbf{1}_{\{T_1 > t\}} \exp\left(\theta c^+ + \zeta \Sigma_2 + \xi\theta^q \sum_{k=2}^{N} T_k^q e^{bT_k\theta}\right)\right] \le \rho\, e^{c_1} < \frac{1}{2}$$

for $\theta > 1$. Here the reader should notice that the choice of ρ depends on the value of ξ (through c_1) which is still to be chosen.

Turning to the second term in (41), it therefore remains to verify that, uniformly in $\rho \in (0, 1/2D)$,

$$\mathbb{E}\left[\mathbf{1}_{\{T_1 \le t\}} \exp\left(\theta c^+ + \zeta \Sigma_2 + \xi\theta^q \Sigma_q e^{bT_1\theta} - \xi\theta^q e^{b\theta}\right)\right] \le \frac{1}{2}$$

for $\theta > 1$, t as chosen above, and a suitable $\xi > 0$. For this to be true, one can take ξ such that

$$\sup_{\theta > 1} \exp\left(\theta c^+ + \zeta\|\Sigma_2\|_\infty + \xi\theta^q e^{b\theta}\left(e^{-b\theta(1-t)} - 1\right)\right) \le \frac{1}{2},$$

(recall $\|\Sigma_q\|_\infty \le 1$), and this is indeed possible because (regardless of the value of ρ)

$$b\theta(1-t) = b\theta e^{-b\theta}\left(\frac{\rho}{D}\right)^{1/\gamma} \le b\theta e^{-b\theta} \le e^{-1} \le \log 2$$

and $e^{-u} - 1 \le -u/2$ for $0 < u < \log 2$, giving

$$\begin{aligned}\theta c^+ + \zeta \|\Sigma_2\|_\infty + \xi\theta^q e^{b\theta}\left(e^{-b\theta(1-t)} - 1\right) &\le \theta c^+ + \zeta \|\Sigma_2\|_\infty - e^{b\theta}\frac{\xi b\, \theta^{q+1}}{2} \\ &= \theta c^+ + \zeta \|\Sigma_2\|_\infty - \frac{\xi b\, \theta^{q+1}}{2}\end{aligned}$$

with the last bound going to 0 as $\xi \to \infty$, uniformly in $\theta \ge 1$.

Having shown that Φ satisfies (30) on $\mathbb{R}_\geqslant$, Lemma 6.1 provides us with

$$\mathbb{E}e^{\theta W} = \Psi(\theta) \le \Phi(\theta) = \exp\left(\xi\theta^q e^{b\theta}\right) \quad \text{for all } \theta > 1.$$

Picking an arbitrary $\varepsilon \in (0, 1)$, there exists $\theta_0 > 1$ such that

$$\exp\left(\xi\theta^q e^{b\theta}\right) \le e^{(b+\varepsilon)\theta} \quad \text{for all } \theta \ge \theta_0.$$

As a consequence, we obtain

$$\mathbb{P}[W > x] \le e^{-\theta x}\,\mathbb{E}e^{\theta W} \le \exp\left(e^{(b+\varepsilon)\theta} - \theta x\right)$$

for $\theta > \theta_0$ and $x > (b+1)e^{\theta_0(b+1)}$. The minimum in θ on the right-hand side occurs at $\theta = \frac{1}{b+\varepsilon}\log\frac{x}{b+\varepsilon}$, and with this θ we obtain

$$\frac{\log \mathbb{P}[W > x]}{x\log x} \le -\frac{1}{b+\varepsilon} + \frac{1+\log(b+\varepsilon)}{\log x}$$

and thereupon

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}[W > x]}{x\log x} \le -\frac{1}{b} = -\frac{\gamma}{c^+},$$

since $\varepsilon \in (0, 1)$ was arbitrary.

LOWER BOUND. Now assume additionally that (20) holds. For $\varepsilon > 0$ consider

$$N_\varepsilon = \sum_{k=1}^{N} \mathbf{1}_{\{T_k > 1-\varepsilon\}}$$

which decreases to $N_0 = \Sigma_\infty$ as $\varepsilon \to 0$. Under the given assumptions, we can fix $\varepsilon_0 \in (0, 1)$ so small that

- (20) is valid with $\varepsilon = \varepsilon_0$,
- $z := \mathbb{P}[\{T_1 \le 1 - \varepsilon_0, C \ge 0\} \cup \{0 \le C \le c^+ - \varepsilon_0\}] > 0$,
- $\|\max_{2\le k\le N} T_k\|_\infty \le 1 - \varepsilon_0$, and
- $0 < \mathbb{P}[T_1 > 1 - \varepsilon_0] < 1$, hence $N_{\varepsilon_0} \le 1$ a.s. and $0 < \mathbb{E}N_{\varepsilon_0} < 1$.

In the associated weighted branching model as specified in Section 2, define the homogeneous stopping line (see [3, Section 7] for the general definition)

$$\mathcal{T}_\varepsilon := \{vk : T_k(v) \le 1 - \varepsilon \text{ or } C(v) \le c,\ \forall\, uj \prec v : C(u) > c,\ T_j(u) > 1 - \varepsilon\}$$

for $\varepsilon \in (0, \varepsilon_0)$ and $c \in (c^+ - \varepsilon_0, c^+)$. Then the SFPE (4) then implies

$$W \overset{d}{=} \sum_{v\in\mathcal{T}_\varepsilon} L(v)W(v) + \sum_{v\prec\mathcal{T}_\varepsilon} L(v)C(v). \tag{42}$$

With $e_k := (1, 1, \dots, 1) \in \mathbb{N}^k$ for $k \in \mathbb{N}$ and $e_0 := \varnothing$, define the stopping time, consider the stopping time

$$\tau := \inf\{k \geq 0 : T_1(e_k) \leq 1 - \varepsilon \text{ or } C(e_k) \leq c\}$$

along the leftmost path in the given weighted branching tree. Plainly, τ has a geometric distribution, viz.

$$\mathbb{P}[\tau = k] = \mathbb{P}[T_1 \leq 1 - \varepsilon \text{ or } C \leq c]\, \mathbb{P}[T_1 > 1 - \varepsilon,\ C > c]^k \quad \text{for } k \geq 0,$$

and we note for later use that, by (20),

$$\begin{aligned}\mathbb{P}[\tau = k,\ C(e_k) \geq 0] &= \mathbb{P}[\{T_1 \leq 1 - \varepsilon, C \geq 0\} \cup \{0 \leq C \leq c\}] \\ &\quad \times \mathbb{P}[T_1 > 1 - \varepsilon,\ C > c]^k \\ &\geq z(\kappa d' \varepsilon^{\gamma})^k \quad \text{for } k \geq 0,\end{aligned} \tag{43}$$

where $\kappa := \mathbb{P}[C > c] > 0$. Next, observe that

$$\mathcal{T}_\varepsilon = \{e_{\tau+1}\} \cup \bigcup_{j=0}^{\tau} \{e_k j : 1 \leq j \leq N(e_k)\},$$

$$\{v \prec \mathcal{T}_\varepsilon\} = \{e_k : 0 \leq k \leq \tau\},$$

and $C(v) > c$ for $k < \tau$, hence

$$\sum_{v \prec \mathcal{T}_\varepsilon} L(v)C(v) \geq \sum_{k=0}^{\tau-1} L(e_k)C(e_k) > \sum_{j=0}^{\tau-1} (1-\varepsilon)^j c = \frac{1-(1-\varepsilon)^\tau}{\varepsilon}\, c. \tag{44}$$

Define the event

$$\mathfrak{X} := \{W(v) \geq 0 \text{ for all } v \in \mathcal{T}_\varepsilon\},$$

and further $r := \mathbb{P}[W \geq 0] > 0$ and $f(s) := \mathbb{E}[s^N]$ for $s \in [0, 1]$. On the event $\{\tau = k,\ C(e_k) \geq 0\} \cap \mathfrak{X}$, we have

$$|\{v < \mathcal{T}_\varepsilon\}| = k + 1, \qquad |\mathcal{T}_\varepsilon| \leq 1 + \sum_{j=0}^{k-1} N(e_j),$$

and

$$\sum_{v \in \mathcal{T}_\varepsilon} L(v)W(v) + \sum_{v \prec \mathcal{T}_\varepsilon} L(v)C(v) > \sum_{j=0}^{k-1} (1-\varepsilon)^j c = \frac{1-(1-\varepsilon)^k}{\varepsilon}\, c.$$

Moreover, using (43),

$$\begin{aligned}\mathbb{P}[\{\tau = k,\ C(e_k) \geq 0\} \cap \mathfrak{X}] &\geq z\kappa\, d' \varepsilon^k\, \mathbb{E}\left[r^{1+\sum_{j=0}^{k-1} N(e_j)} \right] \\ &= zr(\kappa d' \varepsilon^{\gamma})^k f(r)^k\end{aligned}$$

and therefore, in view of (42) and (44),

$$\mathbb{P}\left[\frac{W}{c} > \frac{1-(1-\varepsilon)^k}{\varepsilon} \right] \geq zr(\kappa d' \varepsilon^{\gamma})^k f(r)^k$$

for all $k \in \mathbb{N}_0$. Setting $a := \kappa d' f(r)$, this further yields

$$\mathbb{P}\left[\frac{W}{c} > \frac{1-(1-\varepsilon)^y}{\varepsilon}\right] \geq zr(a\varepsilon^\gamma)^{y+1} \tag{45}$$

for all $y \in \mathbb{R}_\geqslant$. Now let $x \geq c$. Then we may choose, for some $\delta \in (0, \varepsilon)$,

$$\varepsilon := \frac{\delta c}{x} \quad \text{and} \quad y := \frac{\log(1-\delta)}{\log(1-\varepsilon)} = \frac{\log(1-\delta)}{\log(1-\delta c/x)}$$

to further infer from (45)

$$\begin{aligned} \log \mathbb{P}[W > x] &\geq \log(zr) + \left(\frac{\log(1-\delta)}{\log(1-\delta c/x)} + 1\right) \log(a\varepsilon^\gamma) \\ &= \log(zr) - \left(\frac{\log(1-\delta)}{\log(1-\delta c/x)} + 1\right) \left(\gamma \log x - \log(a(\delta c)^\gamma)\right). \end{aligned}$$

Keeping δ fixed and letting x tend to ∞, we have $\log(1-\delta c/x) \simeq -\delta c/x$ and so

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}[W > x]}{x \log x} \geq \frac{\gamma \log(1-\delta)}{\delta c} \geq \frac{\gamma \log(1-\delta)}{\delta(c^+ - \varepsilon_0)}.$$

This being true for any fixed ε_0 and δ sufficiently small, we finally arrive at the desired conclusion

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}[W > x]}{x \log x} \geq -\frac{\gamma}{c^+}$$

by first letting $\delta \downarrow 0$, giving $\frac{\log(1-\delta)}{\delta} \to -1$, and then $\varepsilon_0 \downarrow 0$ (which implies $c \to c^+$). $\square$

Acknowledgments

The main part of this work was done while the second author was visiting the Institute of Mathematical Statistics at Münster in September 2014 and February 2015. He gratefully acknowledges financial support and hospitality. The authors would also like to thank two anonymous referees for their constructive suggestions that helped improving the presentation of this paper.

The first author was partially supported by the Deutsche Forschungsgemeinschaft (SFB 878). The second author was partially supported by the National Science Centre, Poland (Sonata Bis, grant number DEC-2014/14/E/ST1/00588).

References

[1] D.J. Aldous, A. Bandyopadhyay, A survey of max-type recursive distributional equations, Ann. Appl. Probab. 15 (2) (2005) 1047–1110.

[2] G. Alsmeyer, The smoothing transform: a review of contraction results, in: Random Matrices and Iterated Random Functions, in: Springer Proc. Math. Stat., vol. 53, Springer, Heidelberg, 2013, pp. 189–228.

[3] G. Alsmeyer, J.D. Biggins, M. Meiners, The functional equation of the smoothing transform, Ann. Probab. 40 (5) (2012) 2069–2105.

[4] G. Alsmeyer, A. Iksanov, U. Rösler, On distributional properties of perpetuities, J. Theoret. Probab. 22 (3) (2009) 666–682.

[5] G. Alsmeyer, D. Kuhlbusch, Double martingale structure and existence of ϕ-moments for weighted branching processes, Münster J. Math. 3 (2010) 163–211.

[6] G. Alsmeyer, U. Rösler, The best constant in the Topchiĭ-Vatutin inequality for martingales, Statist. Probab. Lett. 65 (3) (2003) 199–206.
[7] G. Alsmeyer, U. Rösler, On the existence of ϕ-moments of the limit of a normalized supercritical Galton-Watson process, J. Theoret. Probab. 17 (4) (2004) 905–928.
[8] A. Baltrūnas, D.J. Daley, C. Klüppelberg, Tail behaviour of the busy period of a $GI/GI/1$ queue with subexponential service times, Stochastic Process. Appl. 111 (2) (2004) 237–258.
[9] M. Drmota, Random Trees. An Interplay Between Combinatorics and Probability, Springer-Verlag, Vienna, New York, 2009.
[10] C.M. Goldie, R. Grübel, Perpetuities with thin tails, Adv. Appl. Probab. 28 (2) (1996) 463–480.
[11] F. Guillemin, P. Robert, B. Zwart, AIMD algorithms and exponential functionals, Ann. Appl. Probab. 14 (1) (2004) 90–117.
[12] P. Hitczenko, J. Wesołowski, Perpetuities with thin tails revisited, Ann. Appl. Probab. 19 (6) (2009) 2080–2101. Corrigendum in 20:1177 (2010).
[13] C.A.R. Hoare, Algorithm 64: Quicksort, Commun. Assoc. Comput. Mach. 4 (1961) 321.
[14] C.A.R. Hoare, Quic ksort, Comput. J. 5 (1962) 10–15.
[15] S. Janson, et al., On the tails of the limiting quicksort distribution, Electron. Comm. Probab. 20 (2015).
[16] B. Kołodziejek, Logaritmic tails of sums of products of positive random variables bounded by one. 2015. arXiv:1510:01066.
[17] R. Neininger, L. Rüschendorf, On the internal path length of d-dimensional quad trees, Random Structures Algorithms 15 (1) (1999) 25–41.
[18] R. Neininger, L. Rüschendorf, Analysis of algorithms by the contraction method: additive and max-recursive sequences, in: Interacting Stochastic Systems, Springer, Berlin, 2005, pp. 435–450.
[19] M.D. Penrose, A.R. Wade, Random minimal directed spanning trees and Dickman-type distributions, Adv. Appl. Probab. 36 (3) (2004) 691–714.
[20] M.D. Penrose, A.R. Wade, On the total length of the random minimal directed spanning tree, Adv. Appl. Probab. 38 (2) (2006) 336–372.
[21] U. Rösler, A limit theorem for "Quicksort", RAIRO Inform. Théor. Appl. 25 (1) (1991) 85–100.
[22] U. Rösler, A fixed point theorem for distributions, Stochastic Process. Appl. 42 (2) (1992) 195–214.
[23] U. Rösler, On the analysis of stochastic divide and conquer algorithms, Algorithmica 29 (1–2) (2001) 238–261. Average-case analysis of algorithms (Princeton, NJ, 1998).