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# Iterated random functions and regularly varying tails

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## ABSTRACT

We consider solutions to so-called stochastic fixed point equation  $R \stackrel{d}{=} \Psi(R)$ , where  $\Psi$  is a random Lipschitz function and  $R$  is a random variable independent of  $\Psi$ . Under the assumption that  $\Psi$  can be approximated by the function  $x \mapsto Ax + B$ , we show that the tail of  $R$  is comparable with the one of  $A$ , provided that the distribution of  $\log(A \vee 1)$  is tail equivalent. In particular, we obtain new results for the random difference equation.

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## 1. Introduction

Let  $\{\Psi_n\}_{n \geq 1}$ ,  $\Psi_n: \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of independent identically distributed (iid) random Lipschitz functions. We consider the Markov chain defined by

$$R_{n+1} = \Psi_{n+1}(R_n), \quad n \geq 0,$$

where  $R_0$  is a random variable independent of  $\{\Psi_n\}_{n \geq 1}$ . Under rather mild moment assumptions, the Markov chain  $\{R_n\}_{n \geq 0}$  possesses a unique stationary distribution. Suppose that  $R$  is distributed according to it and let  $\Psi$  be a generic copy of  $\Psi_n$  independent of  $R$ , then necessarily

$$R \stackrel{d}{=} \Psi(R), \quad R \text{ independent of } \Psi,$$

where  $\stackrel{d}{=}$  denotes the equality in distribution. Distributional equations of this form appear in a wide range of problems in applied probability. Beginning from the early nineties iterated function systems of i.i.d. Lipschitz maps (IFS) on a complete metric space have attracted a lot of attention: Alsmeyer [1], Arnold and Crauel [2], Brofferio and Buraczewski [4], Buraczewski and Damek [5], Diaconis and Friedman [12], Duflo [13], Elton [15], Henion and Hervé [23], Mirek [30]. In particular, it seems that modelling them after random difference equation (described below) has been very fruitful, see Alsmeyer [1] and Mirek [30].

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The main example we wish to present here, is the so-called random difference equation occurring whenever  $\Psi$  is just affine transformation, i.e.  $\Psi(x) = Ax + B$ . Then, the recursive formula for the Markov chain in question which, in this special instance, will be denoted by  $\{X_n\}_{n \geq 0}$ , takes the simple form

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \quad n \geq 0,$$

where  $\{(A_n, B_n)\}_{n \geq 1}$  is a sequence of iid two-dimensional random vectors. Here the stochastic equation satisfied by  $X$  is

$$X \stackrel{d}{=} AX + B, \quad X \text{ independent of } (A, B),$$

where  $(A, B)$  is an independent copy of  $(A_n, B_n)$ . It turns out that the stationary solution can be explicitly represented by

$$X \stackrel{d}{=} \sum_{k=0}^{\infty} B_{k+1} \prod_{j=1}^k A_j$$

provided that the series is convergent. The series above can be interpreted as the current value of future payments represented by  $B_{k+1}$  with discount factors represented by  $A_j$  and therefore, it is very often called a perpetuity. Random variables of this form appear also in context of Additive Increase Multiplicative Decrease algorithms [22] or COGARCH processes [28], to name a few. For more detailed discussion on perpetuities and related processes, we refer the reader to recent monographs [6, 24].

From the point of view of applications, the key information about the distribution of  $X$  is its tail asymptotic, that is

$$\mathbb{P}[X > t] \quad \text{as } t \rightarrow \infty.$$

We wish to recall several scenarios, where  $X$  exhibits regularly varying tail. Assume that  $A, B \geq 0$  for the moment. The first example is related to the work of Kesten [25] and Goldie [19], which shows that if, among some additional technical conditions,

$$\mathbb{E}[A^\alpha] = 1, \quad \mathbb{E}[B^\alpha] < \infty \tag{1}$$

for a positive  $\alpha$ , the stationary distribution has power tail, i.e.

$$\mathbb{P}[X > t] \sim c_X t^{-\alpha} \quad \text{as } t \rightarrow \infty, \tag{2}$$

for some implicitly given constant  $c_X > 0$ . Here, and in what follows, for two functions  $g, h$ , by  $g(t) \sim h(t)$  we mean  $\lim_{t \rightarrow \infty} (g(t)/h(t)) = 1$ . The asymptotic (2) follows from the behaviour of  $A$ , more precisely it is a direct consequence of the first assumption in (1). It may as well happen that a heavy tail of  $X$  is caused by a heavy tail of  $B$ . More precisely, the

work of Grincevičius [21] which was later improved by Grey [20] treats the case

$$\mathbb{P}[B > t] \sim t^{-\alpha}L(t), \quad \mathbb{E}[A^\alpha] < 1, \tag{3}$$

where  $\alpha > 0$  and  $L(t)$  is a slowly varying function, that is  $L(ct) \sim L(t)$  for any fixed  $c > 0$ . Then

$$\mathbb{P}[X > t] \sim \frac{1}{1 - \mathbb{E}[A^\alpha]} \mathbb{P}[B > t].$$

Finally, in the recent work, Kevei [26] proves that, if

$$\mathbb{P}[A > t] \sim t^{-\alpha}L(t), \quad \mathbb{E}[A^\alpha] < 1, \quad \mathbb{E}[B^{\alpha+\varepsilon}] < \infty,^1 \tag{4}$$

for  $\alpha, \varepsilon > 0$  then, provided that  $L$  satisfies some convolution condition,

$$\mathbb{P}[X > t] \sim \frac{\mathbb{E}[X^\alpha]}{1 - \mathbb{E}[A^\alpha]} \mathbb{P}[A > t].$$

One important feature the scenarios (1)–(4) have in common is that either  $A$  or  $B$  contributes significantly to the asymptotic of  $X$ , not both.

In order to get a more detailed information about the structure of the distribution of  $X$ , it is natural to consider the frontiers of the scenarios in question, where both coefficients have influence on the tail asymptotic of  $X$ . First one, situated between (1) and (3) was recently investigated by Damek and Kołodziejek [11], is the case when

$$\mathbb{E}[A^\alpha] = 1, \quad \mathbb{P}[B > t] \sim t^{-\alpha}L(t), \quad \mathbb{E}[B^\alpha] = \infty,$$

which results in

$$\mathbb{P}[X > t] \sim t^{-\alpha}\tilde{L}(t),$$

with some explicitly given slowly varying function  $\tilde{L}$ .

The second one, being the frontier between (3) and (4), is

$$\mathbb{E}[A^\alpha] < 1, \quad \mathbb{P}[A > t] \asymp \mathbb{P}[B > t] \asymp t^{-\alpha}L(t), \tag{5}$$

where  $g(t) \asymp h(t)$  means that

$$0 < \liminf_{t \rightarrow \infty} \frac{g(t)}{h(t)} \leq \limsup_{t \rightarrow \infty} \frac{g(t)}{h(t)} < \infty.$$

Up to our best knowledge, this case was not studied in the literature apart from two specific cases: independent  $A$  and  $B$  treated in [29] and so-called exponential functional of Lévy processes studied in [31]. Our aim is to present a robust approach to treat the scenario (5) and its counterpart for the iterated random functions  $\{R_n\}_{n \geq 0}$  and  $R$ .

We will work under the assumption that  $\Psi$  can be well approximated by the affine transformation, that is

$$|\Psi(x) - Ax| \leq B\phi(x),$$

with

$$\mathbb{P}[B > t] = O(\mathbb{P}[A > t]), \quad \phi \text{ is locally bounded} \quad \text{and} \quad \phi(x) = o(x), \quad \text{as } x \rightarrow \infty$$

and, among some technical assumptions, that for  $\alpha > 0$

$$\mathbb{P}[A > t] \sim t^{-\alpha} L(t).$$

In order to be able to successfully treat the case  $\mathbb{P}[A > t] \asymp \mathbb{P}[B > t]$ , we will need to ensure that the successive iterations of  $\{\Psi_n\}_{n \geq 1}$  are well behaved, i.e. for fixed  $x$ ,

$$\mathbb{P}[\Psi(x) > t] \sim f_+(x)\mathbb{P}[A > t]$$

for some measurable function  $f_+$ . Under the above, will use the approach developed by Grey [20] to show that

$$\mathbb{P}[R > t] \sim \frac{\mathbb{E}[f_+(R)]}{1 - \mathbb{E}[A^\alpha]} \mathbb{P}[A > t] \quad \text{as } t \rightarrow \infty$$

with  $\mathbb{E}[f_+(R)] < \infty$ .

The article is organized as follows. In Section 2, we recall some basic notions related to the class of convolution equivalent distributions. The results followed by some examples are stated in Section 3. The proofs are presented in Section 4. The Appendix contains proofs of some classical properties of the convolution equivalent distributions.

## 2. Convolution equivalent tails

Throughout the paper, we would like to benefit from properties of convolution equivalent distributions. We begin by introducing some basic notation. We will consider distribution, say  $F$ , with right-unbounded support. Write  $F^{*n}$  for  $n$ th-convolution of  $F$  and  $\bar{F}$  for its tail, that is  $\bar{F}(x) = 1 - F(x)$ .

**Definition 2.1:** A distribution  $F$  with right-unbounded support contained in  $[0, +\infty)$  is said to be tail equivalent if there exist  $\alpha \geq 0$  such that the following statements hold. For any fixed  $y \in \mathbb{R}$ , as  $t \rightarrow \infty$

$$\bar{F}(t - y) \sim \bar{F}(t)e^{\alpha y}, \tag{6}$$

$m_\alpha(F) = \int e^{\alpha s} F(ds) < \infty$  and moreover

$$\overline{F^{*2}}(t) \sim 2m_\alpha(F)\bar{F}(t). \tag{7}$$

In that case, we will write  $F \in \mathcal{S}(\alpha)$ . By slight abuse of notation, we will write for random variable  $X$ ,  $X \in \mathcal{S}(\alpha)$  whenever its distribution is a member of  $\mathcal{S}(\alpha)$ .

This class was introduced independently by Chistyakov [7] and Chover et al. [8, 9]. The key feature of the class of convolution equivalent distributions is that only the right

tail behaviour is of significance. For this reason, it is natural to work with a wider class of distributions supported on the whole real line  $\mathbb{R}$ .

**Definition 2.2:** We will say that a distribution  $F$  with right-unbounded support contained in  $\mathbb{R}$  is tail equivalent if  $F$  satisfies (6),  $m_\alpha(F) < \infty$  and (7). If this is the case, we will write  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$ .

It is not difficult to see that  $X \in \mathcal{S}_{\mathbb{R}}(\alpha)$  if, and only if  $X$  conditioned on the set  $\{X \geq 0\}$  is in  $\mathcal{S}(\alpha)$ . Next property of distributions from the class  $\mathcal{S}_{\mathbb{R}}(\alpha)$  will be particularly important for us.

**Lemma 2.3:** Assume that  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$ . If for some distributions  $G_1, G_2, \overline{G}_i(t) \sim k_i \overline{F}(t)$ , then

$$\lim_{t \rightarrow \infty} \frac{\overline{G_1 * G_2}(t)}{\overline{F}(t)} = k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1), \tag{8}$$

where  $m_\alpha(G_i) = \int e^{\alpha s} G_i(ds)$ . Moreover,  $k_i > 0$  implies  $G_i \in \mathcal{S}_{\mathbb{R}}(\alpha)$ . If on the other hand

$$\limsup_{t \rightarrow \infty} \frac{\overline{G}_i(t)}{\overline{F}(t)} \leq k_i,$$

then

$$\limsup_{t \rightarrow \infty} \frac{\overline{G_1 * G_2}(t)}{\overline{F}(t)} \leq k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1) \tag{9}$$

and there is a function  $\eta(t) \geq 0, \lim_{t \rightarrow \infty} \eta(t) = 0$  such that for every  $t$

$$\overline{G_1 * G_2}(t) \leq (k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1) + \eta(t)) \overline{F}(t). \tag{10}$$

Similarly, if

$$\liminf_{t \rightarrow \infty} \frac{\overline{G}_i(t)}{\overline{F}(t)} \geq k_i,$$

then

$$\liminf_{t \rightarrow \infty} \frac{\overline{G_1 * G_2}(t)}{\overline{F}(t)} \geq k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1) \tag{11}$$

and for every  $t$

$$\overline{G_1 * G_2}(t) \geq (k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1) - \eta(t)) \overline{F}(t). \tag{12}$$

For completeness reasons, the proof of the above Lemma can be found in the Appendix. In the case of  $\mathcal{S}(\alpha)$ , for (8), one would classically refer to [10].

Condition (7) present in Definition 2.1 seems to be and in fact it is technical. Since  $\mathcal{S}_{\mathbb{R}}(\alpha)$  is the class we are mainly interested in, before we proceed any further we will

present some sufficient conditions for  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$ . As it was proved by Klüppelberg [27] (see Theorem 2.1), for  $\alpha > 0$

$$F \in \mathcal{S}(\alpha) \quad \Leftrightarrow \quad h(x) = e^{\alpha x} \bar{F}(x) \mathbb{1}_{[0, \infty)}(x) \in \mathcal{S}_d, \quad (13)$$

where  $\mathcal{S}_d$  denotes the class of subexponential densities, namely  $h \in \mathcal{S}_d$  if

$$h(x - y) \sim h(x)$$

for any fixed  $y \in \mathbb{R}$  as  $x \rightarrow \infty$ ,  $m_0 = \int_0^\infty h(y) dy < \infty$  and

$$\int_0^x h(x - y)h(y)dy \sim 2m_0h(x).$$

Knowing sufficient conditions for  $h \in \mathcal{S}_d$ , here Theorems 4.15 and 4.16 in [18], we can rewrite those in terms of  $F$  and obtain the next two corollaries.

**Corollary 2.4:** *Assume that  $\bar{F}(t) \sim e^{-\alpha t}K(t)$ , where  $\alpha > 0$ ,  $K(x - y) \sim K(x)$  for any fixed  $y \in \mathbb{R}$  as  $x \rightarrow \infty$ . If one can find a constant  $c > 0$  for which  $K(2x) \geq cK(x)$  for sufficiently large  $x$ , then  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$  provided that  $\int K(t)dt < \infty$ .*

**Corollary 2.5:** *Suppose we have  $\bar{F}(t) \sim e^{-\alpha t}K(t)$  for  $\alpha > 0$ . If  $-\log K(x)$  is eventually concave and one can find a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that*

- $f(x) \leq x/2$  but  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ,
- $K$  if  $f$ -insensitive, i.e.  $K(x - y) \sim K(x)$  as  $x \rightarrow \infty$ , uniformly in  $y \leq f(x)$ ,
- $xK(f(x)) \rightarrow 0$  as  $x \rightarrow \infty$ ,

then  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$  if additionally  $\int K(t)dt < \infty$ .

**Example 2.6:** By Corollary 2.4, if  $\bar{F}(t) \sim ce^{-\alpha t}t^p$  for  $t > t_0$ ,  $\alpha, c > 0$  and  $p < -1$  then  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$ . If on the other hand  $\bar{F}(t) \sim c \exp\{-\alpha t - \beta t^\gamma\}$  for  $\alpha, \beta, c > 0$  and  $\gamma \in (0, 1)$  then again  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$  but this time by Corollary 2.5 with  $f(x) = \log^{1/\gamma}(x)$ .

Other sufficient conditions, going beyond Corollaries 2.5 and 2.4 can be found in [16] and [10].

### 3. Main result

Recall that we are considering a Markov chain given via

$$R_{n+1} = \Psi_{n+1}(R_n), \quad (14)$$

where  $\{\Psi_n\}_{n \geq 1}$  is a sequence of iid random Lipschitz functions,  $\Psi_n: \mathbb{R} \rightarrow \mathbb{R}$  and  $R_0$  is a random variable independent of these functions. Let  $\Psi$  denote a generic element of

$\{\Psi_n\}_{n \geq 1}$ . As argued by Elton [15], under some mild moment assumptions on  $\Psi$ ,  $\{R_n\}_{n \geq 0}$  has a unique stationary distribution which, realized by random variable  $R$ , satisfies

$$R \stackrel{d}{=} \Psi(R), \quad R \text{ independent of } \Psi. \tag{15}$$

Our key assumption concerning the function  $\Psi$  is being Lipschitz with the Lipschitz constant

$$L = \text{Lip}(\Psi) = \sup_{t \neq s} \frac{|\Psi(t) - \Psi(s)|}{|t - s|}$$

satisfying

$$\mathbb{E}[\log L] < 0. \tag{16}$$

We would like to argue that  $R$  possesses a similar tail asymptotic as  $A$  if  $\Psi$  is close to the function  $x \mapsto Ax + B$ . This can be achieved in several ways, for example consider  $\Psi$  of the form

$$\Psi(x) = Ax + \Phi(x) \quad \text{for } x \in \text{supp}\mathcal{L}(R) \tag{17}$$

where  $\mathcal{L}(R)$  denotes the law of  $R$ ,  $A \in \mathbb{R}$ ,  $B \geq 0$  and

$$|\Phi(x)| \leq B\phi(|x|), \quad \mathbb{P}[B \vee |A| > t] = O(\mathbb{P}[A > t]) \tag{18}$$

as  $t \rightarrow \infty$  and

$$\phi \text{ is locally bounded and } \phi(x) = o(x) \text{ as } x \rightarrow \infty. \tag{19}$$

Note that, if  $\text{supp}\mathcal{L}(R)$  unbounded, then necessarily  $|A| \leq L$ . Indeed, writing

$$L \geq \frac{|\Psi(x) - \Psi(0)|}{|x|} = \frac{|Ax + \Phi(x) - \Phi(0)|}{|x|}$$

we notice that

$$\lim_{|x| \rightarrow \infty} \frac{|\Psi(x) - \Psi(0)|}{|x|} = |A|.$$

We will assume that  $\log(A_+)$  satisfies the conditions of Definition 2.2, that is for  $y > 0$

$$\mathbb{P}[A > ty] \sim y^{-\alpha} \mathbb{P}[A > t] \tag{20}$$

as  $t \rightarrow \infty$  and

$$\mathbb{E}[|A|^\alpha] < 1. \tag{21}$$

Denoting by  $A'$  an independent copy of  $A$ , we will also suppose that

$$\mathbb{P}[A_+ A'_+ > t] \sim 2\mathbb{E}[A_+^\alpha] \mathbb{P}[A > t] \tag{22}$$

for some  $\alpha > 0$ . Assume also that for some measurable functions  $f_+, f_- : \mathbb{R} \rightarrow [0, +\infty)$ ,

$$\mathbb{P}[\Psi(y) > t] \sim f_+(y) \mathbb{P}[A > t] \quad \text{for } y \in \text{supp}\mathcal{L}(R), \tag{23}$$

and

$$\mathbb{P}[\Psi(y) < -t] \sim f_-(y) \mathbb{P}[A > t] \quad \text{for } y \in \text{supp}\mathcal{L}(R). \tag{24}$$

Lastly, define

$$\mu_+ = \mathbb{E}[A^\alpha \mathbf{1}_{\{A > 0\}}] \quad \text{and} \quad \mu_- = \mathbb{E}[|A|^\alpha \mathbf{1}_{\{A < 0\}}]$$

**Theorem 3.1:** Assume that  $\Psi$  is a Lipschitz function satisfying (16)–(22). Then the Markov chain  $\{R_n\}_{n \geq 0}$  converges weakly to  $R$ , which is a unique solution to (15). Moreover

$$\mathbb{P}[|R| > t] = O(\mathbb{P}[A > t]).$$

Furthermore,

(a) Suppose additionally that  $\Psi$  is non-decreasing and  $\mathbb{P}[A > t, \Phi(1) < -t] = o(\mathbb{P}[A > t])$  then

$$\mathbb{P}[|R| > t] \asymp \mathbb{P}[A > t].$$

(b) Suppose that (23) and (24) hold. Then

$$\mathbb{P}[R > t] \sim \frac{(1 - \mu_+) \mathbb{E}[f_+(R)] + \mu_- \mathbb{E}[f_-(R)]}{(1 - \mu_+ - \mu_-)(1 - \mu_+ + \mu_-)} \mathbb{P}[A > t]. \quad (25)$$

One novelty of our result is that it allows the non-linear term in  $\Psi$  to have a substantial contribution.

**Example 3.2:** Assume  $A, B \geq 0$  and that (20)–(21) hold and moreover that

$$\mathbb{P}[B > t] \sim c_B \mathbb{P}[A > t], \quad \mathbb{P}[A > t, B > t] = o(\mathbb{P}[A > t]),$$

for some  $c_B \geq 0$ . Then for  $X$  solving

$$X \stackrel{d}{=} AX + B, \quad X \text{ independent of } (A, B),$$

one has, as  $t \rightarrow \infty$ ,

$$\mathbb{P}[X > t] \sim \frac{\mathbb{E}[X_+^\alpha] + c_B}{1 - \mathbb{E}[A^\alpha]} \mathbb{P}[A > t].$$

It turns out that in the case when  $\mathbb{P}[A > t] \sim \mathbb{P}[B > t]$ , the knowledge only of marginals of  $A$  and  $B$  is insufficient to determine  $\mathbb{P}[X > t]$ . We note that the example in the same vein as the one presented in [14].

**Example 3.3:** We wish to compare the tails of  $X$  with two types of input, i.e. two different vectors  $(A, B)$  with the same marginals. Take any positive random variable  $Z$  such that  $\log Z \in \mathcal{S}(2)$  and denote  $\mu = \mathbb{E}[Z]$ ,  $\sigma = \mathbb{E}[Z^2] < 1$ . First, consider  $A^{(1)}$  and  $B^{(1)}$  independent, both distributed as  $Z$ . Then, by Example 3.2 for

$$X^{(1)} \stackrel{d}{=} A^{(1)}X^{(1)} + B^{(1)} \quad (26)$$

one has

$$\mathbb{P}[X^{(1)} > t] \sim d_1 \mathbb{P}[Z > t].$$

Since the first and the second moments of  $X^{(1)}$  that can be computed explicitly using (26), we have

$$d_1 = \frac{2\mu^3 - \mu + 1}{(1 - \mu)(1 - \sigma)^2}.$$

For the second input, consider  $A^{(2)} = B^{(2)}$  with the same distribution as  $Z$ . Then

$$X^{(2)} \stackrel{d}{=} A^{(2)}X^{(2)} + B^{(2)}$$

can be written as

$$X^{(2)} + 1 \stackrel{d}{=} A^{(2)}(X^{(2)} + 1) + 1.$$

Invoking Example 3.2 once again yields

$$\mathbb{P}[X^{(2)} > t] \sim \mathbb{P}[X^{(2)} + 1 > t] \sim d_2 \mathbb{P}[Z > t],$$

where

$$d_2 = \frac{2\mu + \sigma(1 - \mu)}{(1 - \mu)(1 - \sigma)^2}.$$

Summarizing,  $A^{(1)} \stackrel{d}{=} A^{(2)}$ ,  $B^{(1)} \stackrel{d}{=} B^{(2)}$  but the asymptotic of tails of  $X^{(1)}$  and  $X^{(2)}$  are in general different, since  $d_1$  differs from  $d_2$ .

The example above shows, that in order to determine the exact asymptotic of  $X$  in the case when the tails of  $A$  and  $B$  are comparable, we need to have some information regarding the joint distribution of the vector  $(A, B)$ . One example of such information is encrypted in conditions (23) and (24).

#### 4. Proofs

First note, that convergence in (20) is uniform in the following sense:

$$\sup_{y>c} \left( \frac{\mathbb{P}[A > yt]}{\mathbb{P}[A > t]} - y^{-\alpha} \right) \rightarrow 0 \tag{27}$$

as  $t \rightarrow \infty$  for any  $c > 0$ . See for example Bingham et al. [3]. We begin by noting that the convergence of  $\{R_n\}_{n \geq 0}$  follows from the result of Elton [15]. More precisely, note that (16) reads

$$\mathbb{E}[\log(L)] < 0$$

and that (17) and (18) imply that for some  $x_0 \in \mathbb{R}$ ,

$$\begin{aligned} \mathbb{E}[\log|x_0 - \Psi(x_0)|] &\leq \mathbb{E}[\log(1 + |A|)] + \log(1 + |x_0|) + \mathbb{E}[\log(1 + |B|)] \\ &\quad + \log(1 + |\phi(x_0)|) < \infty. \end{aligned}$$

The main result of Elton [15] implies the next proposition.

**Proposition 4.1:** *Assume that  $\Psi$  satisfies*

$$\mathbb{E}[\log^+ L] < \infty, \quad \mathbb{E}[\log L] < 0 \quad \text{and} \quad \mathbb{E}[\log|x_0 - \Psi(x_0)|] < \infty$$

*for some  $x_0 \in \mathbb{R}$ . Then the Markov chain  $\{R_n\}_{n \geq 1}$  converges weakly, to  $R$ , which is a unique solution to (15).*

We will now establish a weak tail equivalence of  $R$  and  $A$ .

**Proposition 4.2:** *Assume that  $\Psi$  satisfies (16)–(22). Then*

$$\mathbb{P}[|R| > t] = O(\mathbb{P}[A > t]).$$

*If we suppose additionally that  $\Psi$  is non-decreasing,  $\mathbb{P}[A > t, \Phi(1) < -t] = o(\mathbb{P}[A > t])$  then*

$$\mathbb{P}[|R| > t] \asymp \mathbb{P}[A > t].$$

**Proof:** For the first claim, take  $\delta > 0$  small enough for

$$a_\delta = \mathbb{E}[(|A| + \delta B)^\alpha](1 - \delta)^{-\alpha} < 1.$$

Next, pick  $t_1$ , for which  $|\phi(t)| \leq \delta|t| + t_1$ . Then it is true that for  $x \in \text{supp}\mathcal{L}(R)$ ,

$$|\Psi(x)| \leq A^*|x| + B^* \quad \text{a.s.}$$

where  $A^* = |A| + \delta B$  and  $B^* = Bt_1$ . Note that by our assumptions  $\mathbb{P}[A^* > t]$  and  $\mathbb{P}[B^* > t]$  are both  $O(\mathbb{P}[A > t])$ . It is true that for any  $t \in \mathbb{R}$ , if  $R$  is independent from  $\{\Psi_n\}_{n \geq 0}$ ,

$$\mathbb{P}[|R| > t] = \mathbb{P}[|\Psi_1(R)| > t] \leq \mathbb{P}[A_1^*|R| + B_1^* > t],$$

which means that

$$|R| \leq_{st} A_1^*|R| + B_1^*,$$

where  $\leq_{st}$  denotes the stochastic order, i.e.  $U \leq_{st} V$  iff  $\mathbb{P}[U > t] \leq \mathbb{P}[V > t]$  for any  $t \in \mathbb{R}$ . Since, due to independence of  $R$  and  $\Psi_2$ , it is also true that  $|R| \leq_{st} A_2^*|R| + B_2^*$ , we can infer by the merit of  $A_1^*$  being positive that

$$|R| \leq_{st} A_1^*A_2^*|R| + A_1^*B_2^* + B_1^*.$$

Inductively, we can show this way that for any  $n \geq 1$ ,

$$|R| \leq_{st} A_1^*A_2^* \dots A_n^*|R| + \sum_{k=0}^{n-1} A_1^*A_2^* \dots A_k^*B_{k+1}^*. \quad (28)$$

Since  $A_1^*A_2^* \dots A_n^*|R|$  converges in probability to 0 as  $n \rightarrow \infty$ , if we pass to the limit in (28) we get

$$|R| \leq_{st} X^* = \sum_{k=0}^{\infty} A_1^*A_2^* \dots A_k^*B_{k+1}^*.$$

From now, we will focus on delivering the bound for the tail of  $X^*$ . The key observation is that

$$X^* \stackrel{d}{=} A^*X^* + B^*, \quad X^* \text{ independent of } (A^*, B^*).$$

which means that  $X^*$  is the unique stationary distribution of the Markov chain given via

$$X_n^* = A_n^*X_{n-1}^* + B_n^* \quad n \geq 1,$$

where  $X_0^*$  is independent of the sequence of iid two-dimensional random vectors  $\{(A_n^*, B_n^*)\}_{n \geq 0}$ . By Proposition 4.1, the  $X_n^*$  converges weakly to  $X^*$  for any choice of  $X_0^*$ . From

here, we will follow an idea presented previously by Grey [20]. Consider  $Y = TA' \mathbb{1}_{\{A' > y\}}$ , where  $A'$  is an independent copy of  $A$  and  $T, y$  are some large constants. We have

$$\mathbb{P}[Y > t] \sim T^\alpha \mathbb{P}[A > t] \quad \text{and} \quad \mathbb{E}[Y^\alpha] = T^\alpha \mathbb{E}[A^\alpha \mathbb{1}_{\{A > y\}}].$$

By Lemma 2.3, we can write for some constant  $c = c(\delta, t_1)$ ,

$$\begin{aligned} \mathbb{P}[A^*Y + B^* > t] &\leq \mathbb{P}[A^*Y > (1 - \delta)t] + \mathbb{P}[B^* > \delta t] \\ &\leq (a_\delta T^\alpha + c\mathbb{E}[Y^\alpha](1 - \delta)^{-\alpha} + c + cT^\alpha o(1)) \mathbb{P}[A > t] \\ &\leq (a_\delta T^\alpha + cT^\alpha \mathbb{E}[A^\alpha \mathbb{1}_{\{A > y\}}])(1 - \delta)^{-\alpha} + c + cT^\alpha o(1) \mathbb{P}[A > t]. \end{aligned}$$

First, pass with  $t \rightarrow \infty$  and get

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[A^*Y + B^* > t]}{\mathbb{P}[A > t]} \leq a_\delta T^\alpha + cT^\alpha \mathbb{E}[A^\alpha \mathbb{1}_{\{A > y\}}](1 - \delta)^{-\alpha} + c.$$

For large  $T$  and an appropriate choice of  $y = y(T)$ , we can ensure  $c\mathbb{E}[A^\alpha \mathbb{1}_{\{A > y\}}](1 - \delta)^{-\alpha} < 1 - a_\delta$  and obtain

$$a_\delta T^\alpha + cT^\alpha \mathbb{E}[A^\alpha \mathbb{1}_{\{A > y\}}](1 - \delta)^{-\alpha} + c < T^\alpha.$$

This results in

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[A^*Y + B^* > t]}{\mathbb{P}[A > t]} < \lim_{t \rightarrow \infty} \frac{\mathbb{P}[Y > t]}{\mathbb{P}[A > t]}.$$

Whence we can pick  $t_0$ , such that for  $t > t_0$

$$\mathbb{P}[Y > t] \geq \mathbb{P}[A^*Y + B^* > t].$$

Define the law of r. v.  $X_0^*$  via

$$\mathbb{P}[X_0^* > t] = \mathbb{P}[Y > t \mid Y > t_0], \quad \text{for } t \in \mathbb{R}.$$

Then for any  $t \in \mathbb{R}$

$$\mathbb{P}[X_0^* > t] \geq \mathbb{P}[A^*X_0^* + B^* > t].$$

To see that this is in fact true, consider two possibilities, first of which is  $t > t_0$ . Then

$$\begin{aligned} \mathbb{P}[A^*X_0^* + B^* > t] &= \mathbb{P}[A^*Y + B^* > t \mid Y > t_0] \\ &\leq \frac{\mathbb{P}[A^*Y + B^* > t]}{\mathbb{P}[Y > t_0]} \leq \frac{\mathbb{P}[Y > t]}{\mathbb{P}[Y > t_0]} = \mathbb{P}[X_0^* > t]. \end{aligned}$$

For  $t < t_0$ ,  $\mathbb{P}[X_0^* > t] = 1$  so that  $\mathbb{P}[A^*X_0^* + B^* > t] \leq \mathbb{P}[X_0^* > t]$  is trivial. Now, inductively we can write for any  $n \geq 1$ , since  $A^* \geq 0$ ,

$$\mathbb{P}[X_{n+1}^* > t] = \mathbb{P}[A_{n+1}^*X_n^* + B_{n+1}^* > t] \leq \mathbb{P}[A^*X_0^* + B^* > t] \leq \mathbb{P}[X_0^* > t].$$

This completes the proof of the upper bound since

$$\mathbb{P}[R > t] \leq \mathbb{P}[X^* > t] \leq \mathbb{P}[X_0^* > t] \sim \frac{T^\alpha}{\mathbb{P}[Y > t_0]} \mathbb{P}[A > t].$$

For the lower bound, just note that if  $R \geq 1$  then  $\Psi(R) \geq \Psi(1)$  and so

$$\mathbb{P}[R > t] = \mathbb{P}[\Psi(R) > t] \geq \mathbb{P}[\Psi(1) > t, R \geq 1] = \mathbb{P}[R \geq 1] \mathbb{P}[\Psi(1) > t],$$

where

$$\begin{aligned} \mathbb{P}[\Psi(1) > t, A > 0] &= \mathbb{P}[A + \Phi(1) > t] \geq \mathbb{P}[A > 2t] - \mathbb{P}[A > 2t, \Phi(1) < -t] \\ &\geq \mathbb{P}[A > 2t] - \mathbb{P}[A > t, \Phi(1) < -t] \sim 2^{-\alpha} \mathbb{P}[A > t]. \end{aligned}$$

■

After establishing  $\mathbb{P}[|R| > t] = O(\mathbb{P}[A > t])$  it is relatively easy to get the exact asymptotic, provided that one is equipped with (23) and (24). Note that due to the bound

$$f_{\pm}(x) \leq C_{\pm}(|x|^{\alpha} + 1),$$

we know that by the merit of the last proposition,

$$\mathbb{E}[f_{\pm}(R)] < \infty.$$

**Lemma 4.3:** *Suppose (16)–(24) are satisfied. Denote*

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[R > t]}{\mathbb{P}[A > t]} = D_+$$

and

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[R < -t]}{\mathbb{P}[A > t]} = D_-.$$

Then for  $(\Psi, A, B, \Phi)$  independent of  $R$  we have

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[\Psi(R) > t]}{\mathbb{P}[A > t]} \leq \mathbb{E}[|A|^{\alpha} \mathbf{1}_{\{A > 0\}}] D_+ + \mathbb{E}[|A|^{\alpha} \mathbf{1}_{\{A < 0\}}] D_- + \mathbb{E}[f_+(R)]$$

and

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[\Psi(R) < -t]}{\mathbb{P}[A > t]} \leq \mathbb{E}[|A|^{\alpha} \mathbf{1}_{\{A > 0\}}] D_- + \mathbb{E}[|A|^{\alpha} \mathbf{1}_{\{A < 0\}}] D_+ + \mathbb{E}[f_-(R)].$$

**Proof:** The asymptotic of both probabilities  $\mathbb{P}[\Psi(R) > t]$  and  $\mathbb{P}[\Psi(R) < -t]$  can be treated in the same fashion. Whence, we will consider only the first one. Pick  $\delta, \varepsilon \in (0, 1)$

and large  $\gamma > 0$ . Decompose the probability of interest in the following fashion:

$$\begin{aligned} \mathbb{P}[\Psi(R) > t] &= \mathbb{P}[\Psi(R) > t, |R| \leq \gamma] + \mathbb{P}[\Psi(R) > t, R > \gamma] + \mathbb{P}[\Psi(R) > t, R < -\gamma] \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For the first term, write

$$I_1 = \mathbb{P}[\Psi(R) > t, |R| \leq \gamma] = \int_{[-\gamma, \gamma]} \mathbb{P}[\Psi(y) > t] \mathbb{P}[R \in dy].$$

Take  $t_0$  such that  $\phi(t) \leq \delta t + t_0$ . Since (27) holds with  $c = \gamma^{-1}$ , we can find a constant  $D = D(\gamma, \delta)$  such that for  $\mathcal{L}(R)$ - a.a.  $|y| \leq \gamma$

$$\mathbb{P}[|\Psi(y)| > t] \leq \mathbb{P}[|A||y| + B(\delta|y| + t_0) > t] \leq D(|y|^\alpha + 1)\mathbb{P}[A > t].$$

Whence, by the dominated convergence theorem, we are allowed to infer that

$$\lim_{t \rightarrow \infty} \frac{I_1}{\mathbb{P}[A > t]} = \mathbb{E}[f_+(R)\mathbb{1}_{\{|R| \leq \gamma\}}] \leq \mathbb{E}[f_+(R)].$$

Since  $\Psi(R) = AR + \Phi(R) \leq AR + B\phi(|R|)$ , to treat the second term write

$$\begin{aligned} I_2 &\leq \mathbb{P}[(A + \delta B)R + Bt_0 > t, R > \gamma] \\ &\leq \mathbb{P}[A^*R > (1 - \varepsilon)t, R > \gamma] + \mathbb{P}[B^* > \varepsilon t, R > \gamma] = J_1 + J_2, \end{aligned}$$

where  $A^* = (A + \delta B)$  and  $B^* = Bt_0$ . Here, we have for some constant  $c$ ,

$$\limsup_{t \rightarrow \infty} \frac{J_2}{\mathbb{P}[A > t]} \leq c\varepsilon^{-\alpha}\mathbb{P}[R > \gamma].$$

For  $J_1$ , let  $R_\gamma = R\mathbb{1}_{\{R > \gamma\}}$  and so

$$J_1 = \mathbb{P}[A^*R_\gamma > (1 - \varepsilon)t, A^* > 0].$$

Notice that both  $A^*$  and  $R_\gamma$  satisfy assumptions of Lemma 2.3. Indeed,

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}[R_\gamma > t]}{\mathbb{P}[A > t]} = D_+, \quad \limsup_{t \rightarrow \infty} \frac{\mathbb{P}[A^* > t]}{\mathbb{P}[A > t]} = C_+^*(\delta) = C_+^* < \infty.$$

By an appeal to Lemma 2.3, we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}[(A + \delta B)R > (1 - \varepsilon)t, R > \gamma]}{\mathbb{P}[A > t]} &\leq (D_+ \mathbb{E}[(A^*)^\alpha \mathbb{1}_{\{A^* > 0\}}] \\ &\quad + C_+^* \mathbb{E}[R_\gamma^\alpha])(1 - \varepsilon)^{-\alpha}. \end{aligned}$$

Finally, we treat  $I_3$  in exactly the case fashion as  $I_2$  and arrive at

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}[(A + \delta B)R > (1 - \varepsilon)t, R < -\gamma]}{\mathbb{P}[A > t]} &\leq (D_- \mathbb{E}[|A^*|^\alpha \mathbb{1}_{\{A^* < 0\}}] \\ &\quad + C_-^* \mathbb{E}[|R_{-\gamma}|^\alpha])(1 - \varepsilon)^{-\alpha}, \end{aligned}$$

where  $R_{-\gamma} = R\mathbb{1}_{\{R < -\gamma\}}$  and  $C_-^* = C_-^*(\delta) = \limsup_{t \rightarrow \infty} \frac{\mathbb{P}[A^* < -t]}{\mathbb{P}[A > t]}$ . This constitutes

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\mathbb{P}[\Psi(R) > t]}{\mathbb{P}[A > t]} &\leq \mathbb{E}[f_+(R)] + (D_+ \mathbb{E}[(A^*)^\alpha \mathbb{1}_{A^* > 0}] + C_+^* \mathbb{E}[R_\gamma^\alpha]) (1 - \varepsilon)^{-\alpha} \\ &\quad + (D_- \mathbb{E}[|A^*|^\alpha \mathbb{1}_{A^* < 0}] + C_-^* \mathbb{E}[|R_{-\gamma}|^\alpha]) (1 - \varepsilon)^{-\alpha}, \end{aligned}$$

Take  $\gamma \rightarrow \infty$  then and  $\delta, \varepsilon \rightarrow 0$  to obtain the claim.  $\blacksquare$

Using the same decompositions and Fatou's Lemma instead of the dominated convergence theorem we also have a Lemma corresponding to the lower limits.

**Lemma 4.4:** *Suppose (16)–(24) are satisfied. Denote*

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{P}[R > t]}{\mathbb{P}[A > t]} = d_+$$

and

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{P}[R < -t]}{\mathbb{P}[A > t]} = d_-.$$

Then for  $(\Psi, A, B, \Phi)$  independent of  $R$  we have

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{P}[\Psi(R) > t]}{\mathbb{P}[A > t]} \geq \mathbb{E}[|A|^\alpha \mathbb{1}_{\{A > 0\}}] d_+ + \mathbb{E}[|A|^\alpha \mathbb{1}_{\{A < 0\}}] d_- + \mathbb{E}[f_+(R)]$$

and

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{P}[\Psi(R) < -t]}{\mathbb{P}[A > t]} \geq \mathbb{E}[|A|^\alpha \mathbb{1}_{\{A > 0\}}] d_- + \mathbb{E}[|A|^\alpha \mathbb{1}_{\{A < 0\}}] d_+ + \mathbb{E}[f_-(R)].$$

**Proof of Theorem 3.1:** In view of Propositions 4.1 and 4.2, only (25) needs to be proved. Denote

$$\mu_\pm = \mathbb{E}[A_\pm^\alpha], \quad \xi_\pm = \mathbb{E}[f_\pm(R)].$$

The fact that  $R$  satisfies (15) combined with Lemmas 4.4 and 4.3 gives us

$$D_+ + D_- \leq \frac{\xi_+ + \xi_-}{1 - \mu_+ - \mu_-}$$

and

$$d_+ + d_- \geq \frac{\xi_+ + \xi_-}{1 - \mu_+ - \mu_-}.$$

Since  $D_+ \geq d_+$  and  $D_- \geq d_-$ , the two inequalities above imply that  $D_+ = d_+$  and  $D_- = d_-$ . Thus, another appeal to Lemmas 4.4 and 4.3 yields

$$D_+ = \mu_+ D_+ + \mu_- D_- + \xi_+$$

$$D_- = \mu_+ D_- + \mu_- D_+ + \xi_-.$$

Since this system can be solved explicitly, this proves our theorem.  $\blacksquare$

## Note

1. Plus some additional technical assumptions.

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## Appendix

Here, we gathered some facts related to the classes  $\mathcal{S}(\alpha)$  and  $\mathcal{S}_{\mathbb{R}}(\alpha)$  that we used in the article. Recall, that we will consider distribution  $F$  with right-unbounded support. Write  $F^{*n}$  for  $n$ th-convolution of  $F$  and  $\bar{F}$  for its tail, that is  $\bar{F}(x) = 1 - F(x)$ . Before we prove Lemma 2.3, we need the following auxiliary result.

**Lemma A.5:** *Suppose that  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$ . Then for any fixed  $v > 0$ , the limit*

$$\lim_{x \rightarrow \infty} \int_v^{x-v} \frac{\bar{F}(x-y)}{\bar{F}(x)} dF(y) \quad (\text{A1})$$

*exist. Moreover*

$$\lim_{v \rightarrow \infty} \lim_{x \rightarrow \infty} \int_v^{x-v} \frac{\bar{F}(x-y)}{\bar{F}(x)} dF(y) = 0. \quad (\text{A2})$$

**Proof:** We proceed as in the proof of Lemma 2.7 in [17]. Let  $x > 2v$  and let  $X, Y$  be two independent random variables with law  $F$ . Then

$$\begin{aligned} \mathbb{P}(X + Y > x) &= \mathbb{P}(X + Y > x, X \leq v) + \mathbb{P}(X + Y > x, v < X \leq x - v) \\ &\quad + \mathbb{P}(X > x - v, Y > v) + \mathbb{P}(X + Y > x, Y \leq v) \\ &= 2\mathbb{P}(X + Y > x, X \leq v) + \mathbb{P}(X + Y > x, v < X \leq x - v) \\ &\quad + \mathbb{P}(X > x - v, Y > v). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} &= 2 \int_{-\infty}^v \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) \\ &\quad + \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) + \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v). \end{aligned}$$

The third term can be managed quite easily as  $x \rightarrow \infty$ , since by the merit of (6),

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v) = e^{\alpha v} \overline{F}(v).$$

For the same reason, by an appeal to the Lebesgue dominated convergence theorem, we can identify the limit of the first term as

$$\lim_{x \rightarrow \infty} \int_{-\infty}^v \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) = \int_{-\infty}^v e^{\alpha y} dF(y).$$

In view of (7) we are allowed to conclude that

$$2m_\alpha(F) = 2 \int_{-\infty}^v e^{\alpha y} dF(y) + \lim_{x \rightarrow \infty} \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) + e^{\alpha v} \overline{F}(v).$$

This proves our first claim. The second one follows by letting  $v \rightarrow \infty$ . ■

**Lemma A.6:** *Suppose that  $F \in \mathcal{S}_{\mathbb{R}}(\alpha)$  and that  $\overline{G}_i(y) \leq k_i \overline{F}(y)$  for  $y \geq v > 0$  and  $i = 1, 2$ . Then for  $x > 2v > 0$  one has*

$$\int_v^{x-v} \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) \leq k_1 k_2 \left( \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v) + \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) \right) \tag{A3}$$

**Proof:** We have

$$\int_v^{x-v} \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) \leq k_1 \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dG_2(y),$$

which can be bounded further by integrating by parts

$$\begin{aligned} \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dG_2(y) &= - \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} d\overline{G}_2(y) = \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{G}_2(v) \\ &\quad - \frac{\overline{F}(v)}{\overline{F}(x)} \overline{G}_2(x-v) + \int_{x-v}^v \frac{\overline{G}_2(x-y)}{\overline{F}(x)} d\overline{F}(y) \\ &\leq k_2 \left( \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v) + \int_{x-v}^v \frac{\overline{F}(x-y)}{\overline{F}(x)} d\overline{F}(y) \right) \\ &= k_2 \left( \frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v) + \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) \right), \end{aligned}$$

which completes the proof. ■

**Proof of Lemma 2.3:** To prove (8) suppose  $x > 2v$ . As in the proof of Lemma .5, we write

$$\begin{aligned} \frac{\overline{G}_1 * \overline{G}_2(x)}{\overline{F}(x)} &= \int_{-\infty}^v \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) \\ &\quad + \int_{-\infty}^v \frac{\overline{G}_2(x-y)}{\overline{F}(x)} dG_1(y) \\ &\quad + \int_v^{x-v} \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) + \frac{\overline{G}_2(x-v) \overline{G}_1(v)}{\overline{F}(x)}. \end{aligned}$$

The first term, by the Lebesgue dominated convergence theorem, tends to

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_{-\infty}^v \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) &= \lim_{x \rightarrow \infty} \int_{-\infty}^v \frac{\overline{G}_1(x-y)}{\overline{F}(x-y)} \frac{\overline{F}(x-y)}{\overline{F}(x)} dG_2(y) \\ &= k_1 \int_{-\infty}^v e^{\alpha y} dG_2(y). \end{aligned}$$

Note that the second term can be treated in exactly the same fashion. The third one, by Lemmas A.5 and A.6 in negligible, i.e.

$$\lim_{v \rightarrow \infty} \lim_{x \rightarrow \infty} \int_v^{x-v} \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) = 0.$$

Finally, for the last term one has

$$\lim_{x \rightarrow \infty} \frac{\overline{G}_2(x-v)\overline{G}_1(v)}{\overline{F}(x)} = k_2 e^{\alpha v} \overline{G}_1(v).$$

Letting  $v \rightarrow \infty$  we obtain (8). The proof of the fact, that  $G_i \in S_{\mathbb{R}}(\alpha)$  whenever  $k_i > 0$  is exactly the same as that of Lemma 2.7 in [17]. The fact that the distributions there are supported on  $[0, \infty)$  does not play any role. In order to argue in favour of (10), fix  $\varepsilon > 0$  and take  $v > 0$  big enough such that  $\overline{G}_i(t) \leq (k_i + \varepsilon)\overline{F}(t)$ . Let  $x_0 > 2v$  be such that for  $x > x_0$

$$\frac{\overline{F}(x-v)}{\overline{F}(x)} \overline{F}(v) + \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) < \varepsilon.$$

Then in view of (A3)

$$\begin{aligned} \frac{\overline{G}_1 * \overline{G}_2(x)}{\overline{F}(x)} &= \int_{-\infty}^v \frac{\overline{G}_1(x-y)}{\overline{F}(x)} dG_2(y) \\ &\quad + \int_{-\infty}^v \frac{\overline{G}_2(x-y)}{\overline{F}(x)} dG_1(y) + k_1 k_2 \varepsilon. \end{aligned}$$

Keeping  $v$  fixed and taking  $x_0$  possibly larger we have

$$\begin{aligned} \frac{\overline{G}_1 * \overline{G}_2(x)}{\overline{F}(x)} &\leq (k_1 + \varepsilon)m_\alpha(G_2) + (k_2 + \varepsilon)m_\alpha(G_1) + k_1 k_2 \varepsilon \\ &= k_1 m_\alpha(G_2) + k_2 m_\alpha(G_1) + (k_1 k_2 + m_\alpha(G_1) + m_\alpha(G_2))\varepsilon \end{aligned}$$

which shows (10) and (9). (11) and (12) are obtained in the same way. ■