

Precise large deviation estimates for branching process in random environment

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Abstract. We consider a branching process in random environment $\{Z_n\}_{n \geq 0}$, which is a population growth process where individuals reproduce independently of each other with the reproduction law randomly picked at each generation. We describe precise asymptotics of upper large deviations, i.e. $\mathbb{P}[Z_n > e^{\rho n}]$. Moreover in the subcritical case, under the Cramér condition on the mean of the reproduction law, we investigate large deviation estimates for the first passage times of the branching process in question and of its total population size.

Résumé. Nous considérons un processus de branchement dans un environnement aléatoire $\{Z_n\}_{n \geq 0}$, qui est le processus de croissance d'une population où les individus se reproduisent indépendamment les uns des autres avec la loi de reproduction choisie au hasard à chaque génération. Nous décrivons des asymptotiques précises des grandes déviations supérieures, c'est-à-dire de $\mathbb{P}[Z_n > e^{\rho n}]$. De plus dans le cas sous-critique, sous la condition de Cramér pour la moyenne de la loi de reproduction, nous étudions les estimations des grandes déviations pour le premier temps de passage de ce processus de branchement et de la taille totale de sa population.

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1. Introduction

This work concentrates on the branching process in random environment (BPRE) introduced by Smith and Wilkinson [30] as a one of possible generalizations of the classical Galton–Watson process. BPRE models a population growth where individuals reproduce independently of each other with the reproduction law randomly picked at each generation. To put it formally, let \mathcal{Q} be a random measure on the set of non-negative integers \mathbb{N} , more precisely a measurable function taking values in $\mathcal{M} = \mathcal{M}(\mathbb{N})$ the set of all probability measures on \mathbb{N} equipped with the total variation distance. Then a sequence of independent identically distributed (iid) copies of \mathcal{Q} , say $\mathcal{Q} = \{Q_n\}_{n \geq 0}$ is called a random environment. The sequence $Z = \{Z_n\}_{n \geq 0}$ is called a branching process in random environment \mathcal{Q} if $Z_0 = 1$, and

$$(1.1) \quad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_k^n,$$

where given \mathcal{Q} , $\{\xi_k^n\}_{k \geq 1}$ are iid with common distribution Q_n and are independent of Z_n . We would like to mention that, apart from being interesting on its own merits, as pointed out by Kesten, Kozlov and Spitzer [25] BPRE bears some connections to the path structure of nearest neighbour random walk in random environment in one dimension. For a more detailed discussion regarding BPRE itself, we refer the reader to the classical book of Athreya and Ney [5] or a recent monograph of Kersting and Vatutin [23].

Fundamental questions arising after introducing the process Z concern its asymptotic behaviour, which, as it turns out, is mostly determined by the environment in one dimension. Like in the case of the classical Galton–Watson process, the

answer can be expressed solely in terms of the means of the reproduction laws denoted by

$$A_k = \sum_{j=0}^{\infty} j Q_k(j).$$

To be precise, denote by Π_n the quenched expectation of Z_n , i.e. $\Pi_n = \mathbb{E}[Z_n | \mathcal{Q}] = \prod_{k=0}^{n-1} A_k$ and by A a generic copy of A_k . Depending on the behavior of the associated random walk $S_n = \log \Pi_n$ one distinguishes three cases. BPRE Z is called supercritical when S_n drifts to $+\infty$, subcritical when S_n drifts to $-\infty$ and critical, otherwise. It is well known that if $\mathbb{E}[|\log A|] < \infty$, then Z is supercritical (resp. subcritical, critical) if $\mathbb{E}[\log A] > 0$ (resp. $\mathbb{E}[\log A] < 0$, $\mathbb{E}[\log A] = 0$) (Proposition 2.1 in [23]). As a consequence it can be verified that the process Z dies out with probability 1, whenever $\mathbb{E}[\log A] \leq 0$, whereas in the supercritical case the survival probability is positive and the population grows exponentially on the survival set (Theorems 2.1 and 2.4 in [23]). In the supercritical case the parameter $\mathbb{E}[\log A]$ determines the rate of increase of $\log Z_n$ (see Tanny [31] for the corresponding law of large numbers and Huang, Liu [22] for the central limit theorem). The relation between asymptotic behaviour of Z and $\{\Pi_n\}_{n \geq 0}$ goes beyond this phenomenon and was exploited in numerous papers on BPRE [1,2,4,7].

Our aims in this paper are twofold. First, under mild conditions, we describe the sharp asymptotic behaviour of large deviations of Z :

$$(1.2) \quad \mathbb{P}[Z_n > e^{\rho n}]$$

as $n \rightarrow \infty$. This problem got some attention over the past few years resulting in the precise asymptotic behaviour of (1.2) only in the case of geometric reproduction law obtained by Kozlov [26,27] and on the logarithmic scale asymptotic of (1.2) which was described by Bansaye and Berestycki [7], Bansaye and Böinghoff [8], Böinghoff and Kersting [10]. Recently Grama, Liu and Miqueu [19] proved precise large deviation in the sublinear regime. In the present article we prove that the precise asymptotic behaviour of (1.2) takes the form (see Theorem 2.1 below)¹

$$\mathbb{P}[Z_n > e^{\rho n}] \sim \frac{C_1(\rho)}{\sqrt{n}} e^{-I(\rho)n},$$

for explicitly given rate function $I(\rho)$ and for $\rho > \mathbb{E}[\log A]$ in the supercritical case and $\rho > 0$ in the remaining cases. Note that, up to a multiplicative constant, the probability $\mathbb{P}[Z_n > e^{\rho n}]$ possesses the same asymptotics as $\mathbb{P}[\Pi_n > e^{\rho n}]$, as seen from result of Bahadur and Ranga Rao [6] or Petrov [29].

In the second part of the paper we assume that the branching process is subcritical, i.e. $\mathbb{E}[\log A] < 0$. Then the probability of survival up to time n , i.e. $\mathbb{P}[Z_n > 0]$, decays exponentially fast with the exact asymptotic behaviour described by Geiger, Kersting and Vatutin [17] (see also [9]). Although the process usually dies out relatively quickly, the size of the population can be large. Thus, the above results still provide a description of asymptotics of large deviations of Z_n . Further investigation of our techniques reveals that apart from knowing the probability of large deviation, we can also describe the exceedance time (or the first passage time) for Z defined via

$$T_t^Z = \inf\{n \geq 0 | Z_n > t\}.$$

The first passage time T_t^Z was considered in a series of papers by Afanasyev (see e.g. [1–3]) under the Cramér condition

$$(1.3) \quad \mathbb{E}[A^{\alpha_0}] = 1 \quad \text{for some } \alpha_0 > 0.$$

Then as shown by Afanasyev [1], one has as $t \rightarrow \infty$,

$$(1.4) \quad \mathbb{P}[T_t^Z < \infty] = \mathbb{P}\left[\sup_{n \geq 0} Z_n > t\right] \sim ct^{-\alpha_0},$$

for some positive constant c . Moreover, the asymptotic behaviour of T_t^Z conditioned on $T_t^Z < \infty$ is also understood to some extent, since recently Afanasyev [2] proved the law of large numbers

$$(1.5) \quad \frac{T_t^Z}{\log t} | T_t^Z < \infty \xrightarrow{\mathbb{P}} \frac{1}{\rho_0},$$

¹Here and in what follows, we write $f(n) \sim g(n)$ for two functions f and g if $f(n)/g(n) \rightarrow 1$ as $n \rightarrow \infty$.

and the corresponding central limit theorem [3]

$$\frac{T_t^Z - \log t / \rho_0}{\sigma_0 \rho_0^{-3/2} \sqrt{\log t}} \Big| T_t^Z < \infty \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\rho_0 = \mathbb{E}[(\log A)A^{\alpha_0}]$, $\sigma_0 = \mathbb{E}[(\log A)^2 A^{\alpha_0}]$, and $\xrightarrow{\mathbb{P}}$ (resp. \xrightarrow{d}) denotes convergence in probability (resp. in distribution). We study the corresponding precise large deviations of the first passage time. In Theorem 2.2 presented in the next section we establish asymptotics

$$\mathbb{P} \left[\frac{T_t^Z}{\log t} < \frac{1}{\rho} \right] \sim \frac{\mathcal{C}_2(\rho)}{\sqrt{\log t}} t^{-I(\rho)} \quad \text{for } \rho > \rho_0$$

and

$$\mathbb{P} \left[\frac{T_t^Z}{\log t} > \frac{1}{\rho} \right] \sim \frac{\mathcal{C}_3(\rho)}{\sqrt{\log t}} t^{-I(\rho)} \quad \text{for } \rho < \rho_0$$

for some constants $\mathcal{C}_2(\rho), \mathcal{C}_3(\rho) \in (0, \infty)$. In fact we describe the probability that Z exceeds some threshold t precisely at some given moment, that is in Theorem 2.2 we also show that for $\rho \neq \rho_0$,

$$\mathbb{P} \left[T_t^Z = \left\lfloor \frac{\log t}{\rho} \right\rfloor \right] \sim \frac{\mathcal{C}_4(\rho)}{\sqrt{\log t}} t^{-I(\rho)},$$

for some constant $\mathcal{C}_4(\rho) \in (0, \infty)$. As one may expect from (1.4) and (1.5), $I(\rho)$ attains its smallest value at ρ_0 which is $I(\rho_0) = \alpha_0$.

The next process of our interest is $W = \{W_n\}_{n \geq 0}$ describing the total population size up to time n given via

$$W_n = \sum_{k=0}^n Z_k.$$

In Theorem 2.3 we state the large deviation results for W . We also investigate the corresponding first passage time, that is

$$T_t^W = \inf\{n \geq 0 | W_n > t\}$$

as $t \rightarrow \infty$. Then, by the arguments presented by Kesten et al. [25] and Afanasyev [1], see also Lemma 5.2, the following asymptotic formula holds:

$$(1.6) \quad \mathbb{P}[T_t^W < \infty] = \mathbb{P} \left[\sum_{n=0}^{\infty} Z_n > t \right] \sim ct^{-\alpha_0}.$$

After comparing (1.4) and (1.6) one may expect that T_t^W possesses the same normalisation for the law of large numbers and central limit theorem as T_t^Z . This is indeed the case and in Theorem 2.4 we establish the conditional limit theorems for T_t^W as well as a lower deviation result.

The paper is organized as follows. In Section 2 we present a precise statements of our results. In Section 3 we provide preliminaries used in all the proofs for the branching process Z and we present proof of Theorem 2.1. In Sections 4 and 5 we prove our main results for the subcritical case. We will denote the constants by $c_i, i \in \mathbb{N}$ and if a constant is not of our interest, it will be denoted by a generic c , which may change from one line to the other. Constants appearing in our claims will be denoted by \mathcal{C}_i with $i \in \mathbb{N}$.

2. Main results

2.1. Definitions and assumptions

We denote by P_0 a given law on \mathcal{M} , the set of all probability measures on \mathbb{N} . Then the probability measure $P = P_0^{\otimes \mathbb{N}}$ on $\mathcal{M}^{\mathbb{N}}$ defines the law of the environment \mathcal{Q} . Let $(\Gamma, \mathcal{G}) = (\mathbb{N}^{\mathbb{N}}, \mathcal{Bor}(\mathbb{N}^{\mathbb{N}}))$ be the canonical measurable space on which, given the environment \mathcal{Q} , the BPRE Z is defined and denote by $\mathbb{P}_{\mathcal{Q}}$ the corresponding probability measure. Then $(\Gamma \times$

$\mathcal{M}^{\mathbb{N}}, \mathbb{P}$), for $\mathbb{P} = \int \mathbb{P}_{\mathcal{Q}} P(d\mathcal{Q})$, is the total probability space considered below. More precisely for measurable and positive f defined on $\Gamma \times \mathcal{M}^{\mathbb{N}}$, $\int f d\mathbb{P} = \int f(x, \mathcal{Q}) \mathbb{P}_{\mathcal{Q}}(dx) P(d\mathcal{Q})$. We will occasionally abuse the notation and write $\mathbb{P}[\cdot | \mathcal{Q}] = \mathbb{P}_{\mathcal{Q}}[\cdot]$.

As mentioned in the previous section, the multiplicative random walk $\{\Pi_n\}_{n \geq 0}$ plays a crucial role in our analysis, for this reason denote

$$\lambda(\alpha) = \mathbb{E}[A^\alpha] \quad \text{and} \quad \Lambda(\alpha) = \log \lambda(\alpha) = \log \mathbb{E}[A^\alpha].$$

Put $\alpha_\infty = \sup\{\alpha > 0 | \lambda(\alpha) < \infty\}$ and $\alpha_{\min} = \arg \min_{\alpha \geq 0} \lambda(\alpha)$. We assume below that $\lambda(\alpha) < \infty$ for some $\alpha > 0$ and therefore α_∞ is well defined and positive. Then the domain of λ and Λ is $[0, \alpha_\infty)$ and both functions are smooth and convex in $(0, \alpha_\infty)$. Denote by $\rho(\alpha) = \Lambda'(\alpha)$ and $\sigma(\alpha) = \sqrt{\Lambda''(\alpha)}$ standard parameters related to the function Λ . Let $\rho_\infty = \sup\{\rho(\alpha) | \alpha < \alpha_\infty\}$. Recall that the convex conjugate (or the Fenchel–Legendre transform) of Λ is defined by $\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{sx - \Lambda(s)\}$, for $x \in \mathbb{R}$. This rate function appears in the study of large deviations problems for random walks (see e.g. Dembo and Zeitouni [16]). A straightforward calculation yields $\Lambda^*(\rho) = \alpha\rho - \Lambda(\alpha)$ for $\alpha < \alpha_\infty$ and $\rho = \rho(\alpha)$. Lastly, we assume throughout the paper that

$$(2.1) \quad 0 < A < \infty \text{ a.s. and the law of } \log A \text{ is non-lattice,}$$

i.e. the distribution of $\log A$ is not supported on any of the sets $a\mathbb{Z} + b$ for $a > 0$ and $b \in \mathbb{R}$.

2.2. Large deviations of Z

First we state the upper large deviation result for the BPRE Z .

Theorem 2.1. *Assume that (2.1) holds and fix $\alpha \in (0, \alpha_\infty)$ such that $\rho = \rho(\alpha) > 0$. Assume moreover that one of the following conditions holds true:*

$$(H1) \quad \alpha < 1, \mathbb{E}[A_0^{\alpha-1} Z_1 \log^+ Z_1] < \infty \text{ and}$$

$$(2.2) \quad \mathbb{E}[\mathbb{E}[Z_1^{\frac{\alpha}{\alpha-\delta_\alpha}} | \mathcal{Q}]^{\alpha-\delta_\alpha}] < \infty$$

for some $\delta_\alpha \in (0, \alpha)$;

$$(H2) \quad \alpha = 1 \text{ and } \mathbb{E}[Z_1^{1+\delta_1}] < \infty \text{ for some } \delta_1 > 0;$$

$$(H3) \quad \alpha > 1 \text{ and } \lambda(\alpha) > \lambda(1) \text{ and } \mathbb{E}[Z_1^{\alpha+\delta_\alpha}] < \infty \text{ for some } \delta_\alpha > 0.$$

Then

$$\mathbb{P}[Z_n > e^{\rho n}] \sim \frac{C_1(\rho)}{\sqrt{n}} e^{-n\Lambda^*(\rho)} \quad \text{as } n \rightarrow \infty,$$

where the constant $C_1(\rho)$ can be represented by

$$(2.3) \quad C_1(\rho) = \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi}} \lim_{k \rightarrow \infty} \frac{\mathbb{E}Z_k^\alpha}{\lambda(\alpha)^k}$$

and the limit on the right-hand side exists, is finite and positive.

In view of the Bahadur, Ranga Rao and Petrov results stated in Lemma 3.4 below, Theorem 2.1 asserts that upper large deviations probabilities of Z_n and Π_n coincide up to a multiplicative constant. Note that, since the function λ is convex, Theorem 2.1 is valid for arbitrary $\alpha < \alpha_\infty$ both in the supercritical and critical case (of course when the required moment conditions are satisfied). Moreover, if $\alpha \leq 1$, $\alpha < \alpha_\infty$ and $\rho(\alpha) > 0$, both conditions (H1) and (H2) are fulfilled in the subcritical case. The assumption $\lambda(\alpha) > \lambda(1)$ for $\alpha > 1$ can fail only in the strongly subcritical case, when $\lambda'(1) = \mathbb{E}[A \log A] < 0$. Then Kozlov [26,27] and Böinghoff, Kersting [10] observed that the asymptotics of $\mathbb{P}[Z_n > e^{\rho n}]$ and $\mathbb{P}[\Pi_n > e^{\rho n}]$ differ for small values of ρ . In this case also our argument breaks down, see Remark 3.2 for more detailed explanations. Nevertheless even in the strongly subcritical case for large parameters α satisfying (H3) the above theorem is valid.

Finally note that condition (2.2) for $\alpha < 1$, by the conditional Jensen inequality, is guaranteed by a stronger hypothesis $\mathbb{E}[Z_1^{1+\delta'_\lambda}] < \infty$ for $1 + \delta'_\lambda = \alpha/(\alpha - \delta_\lambda)$, which resembles the corresponding assumptions for $\alpha \geq 1$.

2.3. Large deviations of T_t^Z

Now assume that the BPRE Z is subcritical, i.e.

$$\mathbb{E} \log A < 0,$$

which implies the process Z dies out a.s, that is $\mathbb{P}[\lim_{n \rightarrow \infty} Z_n = 0] = 1$. Assume moreover that the condition (1.3) is satisfied. In the upcoming results we will fix a parameter $\rho > 0$ and consider n that depends on t via

$$(2.4) \quad n = n(t) = n_t = \left\lfloor \frac{\log t}{\rho} \right\rfloor$$

and denote additionally

$$\Theta(t) = \frac{\log t}{\rho} - \left\lfloor \frac{\log t}{\rho} \right\rfloor.$$

Under this relation we are able to provide the exact asymptotic of $\mathbb{P}[T_t^Z = n]$. For future reference note that if $\rho = \rho(\alpha)$ for $\alpha \in [\alpha_0, \alpha_\infty)$, then

$$t^{-\alpha} e^{n_t \Lambda(\alpha)} = \lambda(\alpha)^{-\Theta(t)} t^{-\Lambda^*(\rho)/\rho}.$$

Recall that $\rho_0 = \mathbb{E}[A^{\alpha_0} \log A] = \lambda'(\alpha_0)$.

Theorem 2.2. *Let the assumptions of Theorem 2.1 and (1.3) be in force. Take $\alpha \in (0, \alpha_\infty)$ such that $\rho = \rho(\alpha) > 0$ and assume that n and t are related via (2.4). Then, there are constants $\mathcal{C}_2(\rho), \mathcal{C}_3(\rho) \in (0, \infty)$ such that for $\rho > \rho_0$,*

$$(2.5) \quad \mathbb{P}[T_t^Z \leq n] \sim \mathcal{C}_2(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \quad \text{as } t \rightarrow \infty$$

and for $\rho < \rho_0$

$$(2.6) \quad \mathbb{P}[n \leq T_t^Z < \infty] \sim \mathcal{C}_3(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \quad \text{as } t \rightarrow \infty.$$

Moreover for $\rho \neq \rho_0$,

$$(2.7) \quad \mathbb{P}[T_t^Z = n] \sim \mathcal{C}_4(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \quad \text{as } t \rightarrow \infty,$$

for some constant $\mathcal{C}_4(\rho) \in (0, \infty)$.

Up to our best knowledge precise large deviations of T_t^Z of the form (2.7) were not studied in the literature. Result in the same vein but for the sequence of products $\{\Pi_n\}_{n \geq 0}$ was recently obtained by Buraczewski and Maślanka [15] incorporating techniques used previously in work of Buraczewski, Damek and Zienkiewicz [13] in the context of perpetuities (see (2.8) for an example of a perpetuity).

2.4. Large deviations of W and T_t^W

Turning our attention to the total population size, we will approximate W_n by its conditional mean, that is

$$(2.8) \quad R_n = \mathbb{E}[W_n | \mathcal{Q}] = \sum_{k=0}^n \prod_{j=0}^{k-1} A_j = \sum_{k=0}^n \Pi_k.$$

Note that $\{R_n\}_{n \geq 0}$ forms so-called perpetuity sequence and that its structure is more complicated than the one of $\{\Pi_n\}_{n \geq 0}$. Working with perpetuities requires usually more advanced techniques and sometimes this process reveals some new properties (we refer to [12] for more details). Nevertheless, in many aspects the asymptotic behaviour of $\{R_n\}_{n \geq 0}$ is similar to the one of $\{\Pi_n\}_{n \geq 0}$. Thus our main results concerning the total population resemble those stated above for Z , but are slightly weaker (in these settings we were not able to provide sharp pointwise estimates as in (2.7)).

We assume below the Cramér condition (1.3) and formulate large deviations for $\alpha > \alpha_0$ (Theorem 2.3), the law of large numbers and the central limit theorem (Theorem 2.4).

Theorem 2.3. *Under the assumptions of Theorem 2.1 and (1.3) fix $\rho = \rho(\alpha)$ for some $\alpha \in (\alpha_0, \alpha_\infty)$. Then, there exists a constant $C_5(\rho) \in (0, \infty)$ such that*

$$\mathbb{P}[W_n > e^{\rho n}] \sim \frac{C_5(\rho)}{\sqrt{n}} e^{-n\Lambda^*(\rho)} \quad \text{as } n \rightarrow \infty$$

and for any sufficiently big constant D ,

$$\mathbb{P}[W_{\lfloor n - D \log n \rfloor} > e^{\rho n}] = o\left(\frac{1}{\sqrt{n}} e^{-n\Lambda^*(\rho)}\right) \quad \text{as } n \rightarrow \infty.$$

In particular if n and t are related via (2.4), then

$$\mathbb{P}[T_t^W \leq n] \sim C_5(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}},$$

and

$$\mathbb{P}[T_t^W \leq n - D \log n] = o\left(\frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right).$$

From our approach and results stated so far, we know how the deviations of Z and W can occur and more importantly, what is the most probable moment of such deviation. With that knowledge, we are able to derive the corresponding law of large numbers and central limit theorem for T_t^W .

Theorem 2.4. *Suppose the assumptions of Theorem 2.1 and (1.3) are in force. Then*

$$\frac{T_t^W}{\log t} \Big|_{T_t^W < \infty} \xrightarrow{\mathbb{P}} \frac{1}{\rho_0}$$

and

$$\frac{T_t^W - \log t / \rho_0}{\sigma_0 \rho_0^{-\frac{3}{2}} \sqrt{\log t}} \Big|_{T_t^W < \infty} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\rho_0 = \rho(\alpha_0)$ and $\sigma_0 = \sigma(\alpha_0)$. Moreover

$$\mathbb{P}[W_n > e^{\rho_0 n}] \sim C_6(\rho_0) e^{-\alpha_0 \rho_0 n}$$

for some constant $C_6(\rho_0) \in (0, \infty)$.

Observe that the result in Theorem 2.3 is weaker than those in Theorems 2.1 and 2.2. However similar situation takes place when we compare the results concerning $\{\Pi_n\}_{n \geq 0}$ and perpetuities $\{R_n\}_{n \geq 0}$. Then for $\alpha < \alpha_0$ the asymptotic behaviour of the deviation of both processes can be different, see [11, 13].

The same techniques can be used to study BPPE with immigration having applications to random walk in random environment. Similar scheme allows to describe precise large deviations for random walks in random environment [14].

3. Auxiliary results and the proof of Theorem 2.1

3.1. Moments of Z_n

We start with a description of asymptotic behavior of moments of Z_n , as $n \rightarrow \infty$. The following lemma guarantees that in terms of moment generating functions both processes $\{\log(Z_n)\}_{n \geq 0}$ and $\{\log(\Pi_n)\}_{n \geq 0}$ are asymptotically equivalent and proves in particular existence of the limit in (2.3).

Lemma 3.1. *If hypotheses of Theorem 2.1 are satisfied, then the limit*

$$(3.1) \quad c_Z = c_Z(\alpha) = \lim_{n \rightarrow \infty} \frac{\mathbb{E}[Z_n^\alpha]}{\lambda(\alpha)^n}$$

exists and $c_Z(\alpha) \in (0, \infty)$.

Proof. If $\alpha = 1$, the lemma follows trivially since $\mathbb{E}[Z_n] = \lambda(1)^n$. For $\alpha \neq 1$ the claim was essentially proved by Huang and Liu (Theorem 1.3 [22]). However, they worked under different assumptions (the paper was written only for the supercritical case and the authors assumed e.g. $\mathbb{P}[Z_1 = 0] = 0$). For reader convenience we present below complete argument following [22].

Let $U_n = Z_n/\Pi_n$ be the normalized population size. Then the sequence $\{U_n\}_{n \geq 0}$ forms a martingale both under the quenched probability \mathbb{P}_Q and under the annealed law \mathbb{P} with respect to the filtration $\mathcal{F}_n = \sigma(Q_j, \xi_k^j, k \in \mathbb{N}, j \leq n - 1)$. As a positive martingale U_n converges a.s. to some random variable U . By an appeal to Fatou's Lemma $\mathbb{E}U \leq 1$. Of course in the critical and the subcritical case, since the population goes extinct, $U = 0$ a.s. In the supercritical case Tanny [32] proved that U is non-degenerate if and only if

$$(3.2) \quad \mathbb{E}[A_0^{-1} Z_1 \log^+ Z_1] < \infty.$$

Since $\lambda(\alpha) < \infty$, we can define a new probability measure on $\mathcal{M}(\mathbb{N})$ via

$$P_{0,\alpha}(dQ) = \frac{A^\alpha P_0(dQ)}{\lambda(\alpha)}, \quad A = \sum_{j=0}^{\infty} j Q(j).$$

We consider the BPRE Z with respect to the environment distributed according to a new measure $P_\alpha = P_{0,\alpha}^{\otimes \mathbb{N}}$ and we define on $(\Gamma \times \mathcal{M}^{\mathbb{N}})$ the probability measure $\mathbb{P}_\alpha = \int \mathbb{P}_Q P_\alpha(dQ)$. Let \mathbb{E}_α denote the corresponding expectation. Observe that

$$(3.3) \quad \mathbb{E}_\alpha[U_n^\alpha] = \frac{\mathbb{E}[Z_n^\alpha]}{\lambda(\alpha)^n}.$$

Thus to ensure (3.1) it is sufficient to prove existence of the limit $\lim_{n \rightarrow \infty} \mathbb{E}_\alpha[U_n^\alpha]$. Notice that

$$\mathbb{E}_\alpha[\log A] = \frac{\mathbb{E}[A^\alpha \log A]}{\lambda(\alpha)} = \Lambda'(\alpha) = \rho(\alpha) > 0.$$

Therefore, under \mathbb{P}_α the process $\{Z_n\}_{n \geq 0}$ is a supercritical BPRE.

CASE 1: $\alpha < 1$. Since by (H1)

$$\mathbb{E}_\alpha[A_0^{-1} Z_1 \log^+ Z_1] = \frac{1}{\lambda(\alpha)} \mathbb{E}[A_0^{\alpha-1} Z_1 \log^+ Z_1] < \infty,$$

appealing to (3.2) and Tanny [32] we deduce that $\mathbb{P}_\alpha[U > 0] > 0$ and that U_n converges to U in $L^1(d\mathbb{P}_\alpha)$. Therefore

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[Z_n^\alpha]}{\lambda(\alpha)^n} = \lim_{n \rightarrow \infty} \mathbb{E}_\alpha[U_n^\alpha] = \mathbb{E}_\alpha[U^\alpha] > 0.$$

CASE 2: $\alpha > 1$. Now we are going to appeal to Guivarc'h and Liu [20, Theorem 3], who proved that

$$(3.4) \quad 0 < \mathbb{E}_\alpha U^\alpha < \infty \quad \text{if and only if} \quad \mathbb{E}_\alpha[(Z_1/A_0)^\alpha] < \infty \quad \text{and} \quad \mathbb{E}_\alpha[A_0^{1-\alpha}] < 1$$

and both statements are equivalent to $U_n \rightarrow U$ in $L^\alpha(d\mathbb{P}_\alpha)$. Since by (H3)

$$\mathbb{E}_\alpha[(Z_1/A_0)^\alpha] = \lambda(\alpha)^{-1} \mathbb{E}[Z_1^\alpha] < \infty$$

and

$$(3.5) \quad \mathbb{E}_\alpha[A_0^{1-\alpha}] = \lambda(1)/\lambda(\alpha) < 1,$$

the martingale U_n converges to U in $L^\alpha(d\mathbb{P}_\alpha)$ and combining (3.3) with (3.4) we conclude the proof. □

Remark 3.2. The above proof explains the role of the hypothesis $\lambda(1) < \lambda(\alpha)$ in the strongly subcritical case, which appears in (3.5). If this condition is not satisfied in view of the Guivarc'h, Liu result [20] we are not able to ensure finiteness of $\mathbb{E}_\alpha U^\alpha$ and prove (3.1).

Below we prove locally uniform estimates for moments of Z_n and an auxiliary inequality (3.7).

Lemma 3.3. *Under hypotheses of Theorem 2.1 there are $\delta > 0$ and $c = c(\alpha, \delta)$ such that*

$$(3.6) \quad \mathbb{E}[Z_n^s] \leq \begin{cases} c\lambda(s)^n & \text{for any } s \in [\alpha, \alpha + \delta] \text{ if } \alpha \neq 1, \\ cn^8\lambda(s)^n & \text{for any } s \in [1, 1 + \delta] \text{ if } \alpha = 1. \end{cases}$$

Moreover there exists $0 < \gamma < \lambda(\alpha)$ and some constant c such that for any $n \geq 1$,

$$(3.7) \quad \mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^\alpha] \leq c\gamma^n.$$

Proof. STEP 1. PROOF OF (3.6) AND (3.7) FOR $\alpha < 1$. Take any $s \in [\alpha, 1]$, $s < \alpha_\infty$. Since the function $x \mapsto x^s$ is concave, the conditional Jensen inequality entails

$$(3.8) \quad \mathbb{E}[Z_n^s] = \mathbb{E}[\mathbb{E}[Z_n^s | \mathcal{Q}]] \leq \mathbb{E}[\Pi_n^s] = \lambda(s)^n,$$

which proves (3.6) with $c = 1$ and any $\delta < \min\{1 - \alpha, \alpha_\infty - \alpha\}$.

The second part of the lemma for $\alpha < 1$ follows essentially from an estimate which is a consequence the Marcinkiewicz–Zygmund inequality. If $\{X_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables such that $\mathbb{E}X = 0$, $\mathbb{E}|X|^p < \infty$ for some $p \in [1, \infty)$ and $S_n = \sum_{i=1}^n X_i$, then

$$(3.9) \quad \mathbb{E}|S_n|^p \leq C_p \mathbb{E}|X_1|^p \cdot n^{\frac{p}{2} \vee 1}$$

for some universal constant $C_p > 0$ depending only on p . One can easily check that (3.9) follows by applying the Marcinkiewicz–Zygmund inequality and using either the subadditivity of the function $x \mapsto x^{\frac{p}{2}}$ for $p \in [1, 2]$ or the convexity of $x \mapsto x^{\frac{p}{2}}$ for $p > 2$ (see e.g. Theorem I.5.1 in [21]). We also need the following decomposition being an immediate consequence of (1.1):

$$(3.10) \quad Z_n - A_{n-1}Z_{n-1} = \sum_{k=1}^{Z_{n-1}} (\xi_k^{n-1} - A_{n-1}),$$

where $\xi_k^{n-1} - A_{n-1}$, given \mathcal{Q} , are iid with zero mean and independent of Z_{n-1} . Using the decomposition (3.10) in combination with (3.9) yields for any $p \geq 1$,

$$(3.11) \quad \mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^p | \mathcal{Q}, Z_{n-1}] \leq C_p \mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^p | \mathcal{Q}] \cdot Z_{n-1}^{\frac{p}{2} \vee 1}.$$

Since $\rho(\alpha) > 0$, there exists $\varepsilon < \min\{\alpha/2, \delta_\alpha\}$ for δ_α as in condition (H1), such that $\lambda(\alpha - \varepsilon) < \lambda(\alpha)$. Applying the conditional Jensen inequality for the convex function $x \mapsto x^{\frac{1}{\alpha-\varepsilon}}$ and next the inequality (3.9) with $p = \frac{\alpha}{\alpha-\varepsilon}$ (thanks to our choice of ε we have $p = \frac{\alpha}{\alpha-\varepsilon} \in (1, 2)$) we can write

$$\begin{aligned} \mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^\alpha] &\leq \mathbb{E}[\mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^{\frac{\alpha}{\alpha-\varepsilon}} | \mathcal{Q}, Z_{n-1}]^{\alpha-\varepsilon}] \\ &\stackrel{(3.11)}{\leq} C_p^{\alpha-\varepsilon} \mathbb{E}[Z_{n-1}^{\alpha-\varepsilon}] \cdot \mathbb{E}[\mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^{\frac{\alpha}{\alpha-\varepsilon}} | \mathcal{Q}]^{\alpha-\varepsilon}], \end{aligned}$$

where in the last inequality we used independence of $\mathbb{E}[Z_{n-1}^{\alpha-\varepsilon} | \mathcal{Q}]$ and $\mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^{\frac{\alpha}{\alpha-\varepsilon}} | \mathcal{Q}]$ under P . The expectation of the latter is finite and equal for all $n \in \mathbb{N}$ by hypothesis (H1), because (ξ_1^{n-1}, A_{n-1}) has the same distribution as (Z_1, A_0) . Finally appealing to (3.8) and recalling that $\lambda(\alpha - \varepsilon) < \lambda(\alpha)$, we conclude (3.7).

STEP 2: PROOF OF (3.6) FOR $\alpha \in [1, \alpha_\infty)$. Below we will prove a slightly different inequality, saying that for any $\beta \in [1, \alpha + \delta_\alpha) \cap (0, \alpha_\infty)$ (with δ_α as in (H2) and (H3))

$$(3.12) \quad \mathbb{E}[Z_n^\beta] \leq d(\beta) \cdot \begin{cases} \lambda(\beta)^n & \text{if } \lambda(\beta) > \lambda(1), \\ n^{6\beta+1}\lambda(1)^n & \text{if } \lambda(\beta) \leq \lambda(1), \end{cases}$$

for some continuous function d on $[1, \alpha + \delta_\alpha]$. Note that (3.12) entails (3.6). Indeed, this is clear for $\alpha > 1$ (it is sufficient to choose $\delta \in (0, \delta_\alpha)$ and $c = \sup_{\beta \in [\alpha, \alpha + \delta]} d(\beta)$, where finiteness of c follows from continuity of the function d). Whereas for $\alpha = 1$, just recall that $\rho = \rho(1) > 0$, thus $\lambda(s) > \lambda(1)$ for $s > 1$.

To argue in favour of (3.12) first note that for any $\varepsilon, \beta > 0$ and $x, y > 0$,

$$|x + y|^\beta \leq (1 + \varepsilon)|x|^\beta + c(\beta, \varepsilon)|y|^\beta,$$

where $c(\beta, \varepsilon) = (1 - (1 + \varepsilon)^{-\beta^{-1}})^{-\beta} \sim (\varepsilon/\beta)^{-\beta}$ as $\varepsilon \rightarrow 0$. One can easily verify it by considering two cases $|y| \leq ((1 + \varepsilon)^{1/\beta} - 1)|x|$. Combining the above inequality with decomposition $Z_n = Z_{n-1}A_{n-1} + (Z_n - Z_{n-1}A_{n-1})$ we obtain that for any $\varepsilon > 0$,

$$(3.13) \quad \mathbb{E}[Z_n^\beta] \leq (1 + \varepsilon)\lambda(\beta)\mathbb{E}[Z_{n-1}^\beta] + c(\beta, \varepsilon)\mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^\beta].$$

Integrating both sides of inequality (3.11) and recalling independence of $\mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^\beta | \mathcal{Q}]$ and $\mathbb{E}[Z_{n-1} | \mathcal{Q}]$ under P , we obtain

$$(3.14) \quad \mathbb{E}[|Z_n - A_{n-1}Z_{n-1}|^\beta] \leq C_\beta \mathbb{E}[|Z_1 - A_0|^\beta] \mathbb{E}[Z_{n-1}^{\beta^*}],$$

where $\beta^* = \frac{\beta}{2} \vee 1$. (Note that we are not able to deduce here (3.7) because in the subcritical case it can happen that $\rho(\beta^*) < 0$ and thus we cannot refer to Lemma 3.1 and bound $\mathbb{E}[Z_{n-1}^{\beta^*}]$ by $c\lambda(\beta^*)^n$.) Combining (3.13), (3.14) with the inequality $1 + \varepsilon \leq e^\varepsilon$ for $\varepsilon = n^{-2}$ we end up with the following estimate

$$\mathbb{E}[Z_n^\beta] \leq e^{\frac{1}{n^2}} \lambda(\beta) \mathbb{E}[Z_{n-1}^\beta] + c(\beta) n^{2\beta} \mathbb{E}[Z_{n-1}^{\beta^*}],$$

where $c(\beta) = c_0 C_\beta \mathbb{E}[|Z_1 - A_0|^\beta] \beta^\beta$. By iterating this inequality we obtain

$$(3.15) \quad \begin{aligned} \mathbb{E}[Z_n^\beta] &\leq e^{1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}} \lambda(\beta)^n + c(\beta) \sum_{k=1}^n e^{\frac{1}{(n-k+1)^2} + \dots + \frac{1}{n^2}} \lambda(\beta)^{k-1} (n-k+1)^{2\beta} \mathbb{E}[Z_{n-k}^{\beta^*}] \\ &\leq e^2 \lambda(\beta)^n + e^2 c(\beta) \sum_{k=1}^n \lambda(\beta)^k (n-k+1)^{2\beta} \mathbb{E}[Z_{n-k}^{\beta^*}]. \end{aligned}$$

The above estimate is the key step in the proof of (3.12). From now we proceed by induction on m such that $\beta \in (2^m, 2^{m+1}]$, i.e. our induction hypothesis is (3.12).

Inequality (3.12) is obvious for $\beta = 1$. Assume that $\beta \in (1, 2]$, then $\beta^* = 1$ and $\mathbb{E}[Z_{n-k}^{\beta^*}] = \mathbb{E}[Z_{n-k}] = \lambda(1)^{n-k}$. If $\lambda(\beta) > \lambda(1)$, then

$$\mathbb{E}[Z_n^\beta] \leq d(\beta) \lambda(\beta)^n$$

for

$$d(\beta) = e^2 \left[1 + c(\beta) \sum_{k=0}^{\infty} \left(\frac{\lambda(1)}{\lambda(\beta)} \right)^k (k+1)^{2\beta} \right].$$

Otherwise, if $\lambda(\beta) \leq \lambda(1)$,

$$\mathbb{E}[Z_n^\beta] \leq d_2(\beta) n^{2\beta+1} \lambda(1)^n$$

for $d_2(\beta) = e^2(1 + c(\beta))$. Therefore we obtain (3.12) for $\beta \in (1, 2]$.

Assume now that $\beta \in (2^m, 2^{m+1}]$ for $m \geq 1$. We again consider two cases. If $\lambda(\beta) > \lambda(1)$, then by convexity $\lambda(\beta/2) < \lambda(\beta)$ and by the induction hypothesis

$$(3.16) \quad \mathbb{E}[Z_n^{\beta/2}] \leq d(\beta/2) n^{3\beta+1} \max\{\lambda(1), \lambda(\beta/2)\}^n.$$

Since $\beta^* = \beta/2$, combining the above inequality with (3.15) we obtain

$$\mathbb{E}[Z_n^\beta] \leq d(\beta) \lambda(\beta)^n$$

for

$$d(\beta) = e^2 \left[1 + c(\beta) \sum_{k=0}^{\infty} d(\beta/2) k^{3\beta+1} \left(\frac{\max\{\lambda(1), \lambda(\beta/2)\}}{\lambda(\beta)} \right)^k (k+1)^{2\beta} \right].$$

Finally, assume $\lambda(\beta) \leq \lambda(1)$. In this case $\lambda(\beta/2) < \lambda(1)$ and by the induction hypothesis

$$\mathbb{E}[Z_n^{\beta/2}] \leq d(\beta/2) n^{3\beta+1} \lambda(1)^n.$$

Therefore, in view of (3.15) we have

$$\mathbb{E}[Z_n^\beta] \leq d_2(\beta) n^{6\beta+1} \lambda(1)^n$$

for $d_2(\beta) = e^2(1 + c(\beta))$.

STEP 3. PROOF OF (3.7) FOR $\alpha \in [1, \alpha_\infty]$. Observe that for $\alpha > 1$ combining (3.12) with (3.14) we obtain (3.7). Indeed, since $\rho(\alpha) > 0$, $\lambda(\alpha) > \lambda(1)$ and the function λ is convex, one can take $\gamma = \lambda(\alpha_1)$ for some $\alpha_1 < \alpha$ such that $\max\{\lambda(1), \lambda(\alpha^*)\} < \lambda(\alpha_1) < \lambda(\alpha)$, where $\alpha^* = \alpha/2 \vee 1$.

It remains to prove (3.7) for $\alpha = 1$, which can be done applying similar arguments as above. Namely, choose $\varepsilon < \min\{\delta_1, 1\}$ such that $\lambda(1/(1 + \varepsilon)) < \lambda(1)$. Then, applying the conditional Jensen inequality and the inequality (3.11) with $p = 1 + \varepsilon \in (1, 2)$, we have

$$\begin{aligned} \mathbb{E}[|Z_n - A_{n-1} Z_{n-1}|] &\leq \mathbb{E}[\mathbb{E}[|Z_n - A_{n-1} Z_{n-1}|^p | \mathcal{Q}, Z_{n-1}]^{1/p}] \\ &\stackrel{(3.11)}{\leq} C_p^{1/p} \mathbb{E}[\mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^p | \mathcal{Q}]^{1/p} \cdot Z_{n-1}^{1/p}]. \end{aligned}$$

Recall once again that $\mathbb{E}[|\xi_1^{n-1} - A_{n-1}|^p | \mathcal{Q}]$ and $\mathbb{E}[Z_{n-1}^{1/p} | \mathcal{Q}]$ are independent under P . Therefore the last expression can be dominated by

$$C(\mathbb{E}[\mathbb{E}[|\xi_1^{n-1}|^p | \mathcal{Q}]^{1/p}] + \mathbb{E}[A_{n-1}]) \cdot \mathbb{E}[Z_{n-1}^{1/p}] \leq C(\mathbb{E}[Z_1^p]^{1/p} + \lambda(1)) \cdot \mathbb{E}[Z_{n-1}^{1/p}],$$

where for the last inequality we used the Jensen inequality. Finally invoking (3.6) we conclude (3.7) with $\gamma = \lambda(1/(1 + \varepsilon))$ and complete thus proof of the lemma. □

3.2. Large deviations of $\{\Pi_n\}_{n \geq 0}$, $\{Z_n\}_{n \geq 0}$ and of their perturbations

Our starting point is the following well-known classical result describing precise large deviations for random walks due to Bahadur and Ranga Rao [6] and Petrov [29].

Lemma 3.4. *Assume (2.1) and fix an arbitrary $\alpha \in (0, \alpha_\infty)$ such that $\rho = \rho(\alpha) > 0$. If $\{\bar{\delta}_n\}_{n \geq 0}$ is a sequence of non-negative numbers converging to 0, then*

$$(3.17) \quad \mathbb{P}[\Pi_n > e^{n\rho + n\delta_n}] = \frac{(1 + o(1))}{\alpha\sigma(\alpha)\sqrt{2\pi}} \cdot \frac{e^{-n\Lambda^*(\rho)}}{\sqrt{n}} \exp\left\{-\alpha n\delta_n - \frac{n\delta_n^2}{2\sigma^2(\alpha)}(1 + O(|\delta_n|))\right\},$$

uniformly over $\{\delta_n\}_{n \geq 0}$ such that $|\delta_n| \leq \bar{\delta}_n$.

For our purposes we will need a slightly different property with a perturbation of the time parameter (see (3.19) below), which, as we will see, is satisfied by $\{\Pi_n\}_{n \geq 0}$, $\{Z_n\}_{n \geq 0}$ and some other auxiliary sequences of random variables which will be investigated in the sequel.

Definition 3.5. Given $\alpha \in (0, \alpha_\infty)$ such that $\rho = \rho(\alpha) > 0$, we say that a random array $\{V_{n,j}\}_{n,j \geq 0}$ has a property (P_α) with a constant C_V if

1. there is a constant C such that

$$(3.18) \quad \mathbb{P}[V_{n,j} > t] \leq C\lambda(\alpha)^j t^{-\alpha}, \quad n, j \in \mathbb{N}, t > 0;$$

2. for an arbitrary sequence $\{\bar{\delta}_n\}_{n \geq 0}$ of non-negative numbers converging to 0,

$$(3.19) \quad \mathbb{P}[V_{n,n-i_n} > e^{n\rho+n\delta_n}] = (1 + o(1))C_V \cdot \frac{e^{-n\Lambda^*(\rho)}}{\sqrt{n}} e^{-\alpha n\delta_n - i_n\Lambda(\alpha)},$$

uniformly over sequences of integers $\{i_n\}_{n \geq 0}$ and sequences of real numbers $\{\delta_n\}_{n \geq 0}$ such that

$$(3.20) \quad \max\{n^{1/2}|\delta_n|, n^{-1/2}i_n\} \leq \bar{\delta}_n.$$

If the random array $\{V_{n,j}\}_{n,j \geq 0}$ has the property (P_α) and does not depend on the first parameter, then we define random variables $V_j = V_{n,j}$ and say that the sequence $\{V_n\}_{n \geq 0}$ has the property (P_α) .

As we can see the asymptotics (3.19) for the array $V_{n,j} = \Pi_j$ looks like a corollary of (3.17) and this is indeed the case. One can easily prove that the array $\{V_{n,j}\}_{n,j \geq 0}$ (equivalently the sequence $\{\Pi_n\}_{n \geq 0}$) has the property (P_α) (inequality (3.18) follows from the Markov inequality, whereas (3.19) is a consequence of Lemma 3.4; see Lemma 2.4 in [13]).

Define

$$\Pi_{k,n} = \prod_{j=k}^{n-1} A_j, \quad \text{if } k < n \quad \text{and} \quad \Pi_{k,n} = 1, \quad \text{if } k \geq n.$$

To prove our main results we intend to compare Z_n with Π_n , however for technical reasons it is more convenient to compare Π_n and Z_n with auxiliary sequences $Z_{j_n}\Pi_{j_n,n}$ and $Z_{n-j_n}\Pi_{n-j_n,n}$ for $j_n \asymp \log n$, that is for j_n 's for which there exists a constant $C > 0$ such that for all $n \geq 1$,

$$\frac{1}{C} \log n \leq j_n \leq C \log n.$$

The fact that we work with the sequences $Z_{j_n}\Pi_{j_n,n}$ and $Z_{n-j_n}\Pi_{n-j_n,n}$ is the reason to consider arrays rather than sequences in the property (P_α) . Our aim is to prove that all these arrays and sequences have the property (P_α) . For this purpose we need the following lemma:

Lemma 3.6. *Assume (2.1) and fix an arbitrary $\alpha \in (0, \alpha_\infty)$ such that $\rho = \rho(\alpha) > 0$. Let $H = \{H_n\}_{n \geq 0}$ be a sequence of non-negative integer-valued random variables such that*

$$(3.21) \quad \mathbb{E}[H_n^\alpha] \sim c_H \lambda(\alpha)^n$$

for some $c_H = c_H(\alpha) > 0$ and that for some $\varepsilon_0 \in (0, \alpha_\infty - \alpha)$, one can find a constant $c(\alpha, \varepsilon_0)$ such that

$$\mathbb{E}[H_n^s] \leq c(\alpha, \varepsilon_0) n^{c(\alpha, \varepsilon_0)} \lambda(s)^n \quad \text{for all } n \text{ and } s \in [\alpha, \alpha + \varepsilon_0].$$

Let $\{V_n\}_{n \geq 0}$ be a sequence of random variables with the property (P_α) with the constant C_V . Denote by $\bar{V}_{j_n,i}$ a copy of V_{i-j_n} independent of H_{j_n} , for some sequence of integers $\{j_n\}_{n \geq 0}$ such that $j_n \asymp \log n$. Then the array $U_{n,i} = H_{j_n} \bar{V}_{j_n,i}$ has the property (P_α) with the constant $c_H C_V$.

The lemma will be used twice. First for the pair $(H_n, V_n) = (Z_n, \Pi_n)$ and the array $U_{n,i} = Z_{j_n} \Pi_{j_n,i}$. The (P_α) property of the latter will imply that the sequence $\{Z_n\}_{n \geq 0}$ shares this property as well (Lemma 3.7) and this ensures Theorem 2.1. Afterwards, to prove Theorem 2.2, we will invoke again Lemma 3.6, but this time for the pair $(H_n, V_n) = (\Pi_n, Z_n)$ and the array $U_{n,i} = Z_{n-j_n} \Pi_{n-j_n,i}$ (see Lemma 4.1 below).

Proof of Lemma 3.6. Inequality (3.18) for the array $\{U_{n,j}\}_{n,j \geq 0}$ is a consequence of (3.21), the property (P_α) of V_n and the Markov inequality. To prove (3.19) we divide the argument in three steps. Fix an arbitrary sequence $\{\bar{\delta}_n\}_{n \geq 0}$ of non-negative numbers converging to 0. Let $\{i_n\}_{n \geq 0}$ be any sequence of integers and let $\{\delta_n\}_{n \geq 0}$ be any sequence of real numbers satisfying (3.20).

STEP 1: BIG VALUES OF H_{j_n} . We will show that the set where H_{j_n} attains large values is negligible, i.e. for $\beta \in (1/2, 1)$ we have

$$(3.22) \quad \mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} > e^{\rho j_n + j_n^\beta}] = o\left(\frac{e^{-n\Lambda^*(\rho) - \alpha n\delta_n - \Lambda(\alpha)i_n}}{\sqrt{n}}\right).$$

Dividing values of H_{j_n} into geometric intervals and invoking independence of H_{j_n} and $\bar{V}_{j_n, n-i_n}$ we obtain

$$\begin{aligned} & \mathbb{P}[U_{n, n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} > e^{\rho j_n + j_n^\beta}] \\ & \leq \sum_{m \geq 0} \mathbb{P}[H_{j_n} \bar{V}_{j_n, n-i_n} > e^{n\rho+n\delta_n}, e^{\rho j_n + j_n^\beta} e^m < H_{j_n} \leq e^{\rho j_n + j_n^\beta} e^{m+1}] \\ & \leq \sum_{m \geq 0} \mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n - \rho j_n - j_n^\beta} e^{-m-1}] \mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m]. \end{aligned}$$

We will now split the sum into two parts which correspond to $m \leq (\log n)^2$ and $m > (\log n)^2$ respectively.

STEP 1A: $m > (\log n)^2$. Recall the formula for the Fenchel–Legendre transform:

$$(3.23) \quad \Lambda^*(\rho) = \alpha\rho - \Lambda(\alpha).$$

The first factor in the term of the series can be bounded using the inequality (3.18) in the following fashion

$$\mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n} e^{-\rho j_n - j_n^\beta} e^{-m-1}] \leq c e^{-n\Lambda^*(\rho)} e^{-\Lambda(\alpha)j_n - \Lambda(\alpha)i_n - n\alpha\delta_n + \alpha\rho j_n + \alpha j_n^\beta + \alpha m}.$$

In order to treat the second factor, take any $\varepsilon \in (0, \varepsilon_0)$ and write the Taylor expansion $\Lambda(\alpha + \varepsilon) = \Lambda(\alpha) + \rho\varepsilon + \frac{\varepsilon^2}{2}\Lambda''(s)$, for some $s \in [\alpha, \alpha + \varepsilon]$. Taking $c \geq \sup_{s \in [\alpha, \alpha + \varepsilon_0]} \Lambda''(s) + c(\alpha, \varepsilon_0)$ we may write

$$\begin{aligned} (3.24) \quad \mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m] & \leq \mathbb{E}[H_{j_n}^{\alpha+\varepsilon}] e^{-\rho(\alpha+\varepsilon)j_n - (\alpha+\varepsilon)j_n^\beta} e^{-(\alpha+\varepsilon)m} \\ & \leq c j_n^c e^{j_n \Lambda(\alpha+\varepsilon)} e^{-\rho(\alpha+\varepsilon)j_n - (\alpha+\varepsilon)j_n^\beta} e^{-(\alpha+\varepsilon)m} \\ & \leq c j_n^c e^{j_n \Lambda(\alpha) + \varepsilon \rho j_n + c j_n \varepsilon^2} e^{-\rho(\alpha+\varepsilon)j_n - (\alpha+\varepsilon)j_n^\beta} e^{-(\alpha+\varepsilon)m} \\ & = c j_n^c e^{j_n \Lambda(\alpha)} e^{c j_n \varepsilon^2} e^{-\rho \alpha j_n - (\alpha+\varepsilon)j_n^\beta} e^{-(\alpha+\varepsilon)m}. \end{aligned}$$

If we put these two bounds together and sum over $m > (\log n)^2$, we are allowed to infer that

$$\begin{aligned} & \sum_{m > (\log n)^2} \mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n - \rho j_n - j_n^\beta} e^{-m-1}] \mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m] \\ & \leq c e^{-\varepsilon(\log n)^2} \frac{e^{-n\Lambda^*(\rho)}}{1 - e^{-\varepsilon}} j_n^c \exp\{-n\alpha\delta_n - \Lambda(\alpha)i_n - \varepsilon j_n^\beta + c j_n \varepsilon^2\}. \end{aligned}$$

STEP 1B: $m \leq (\log n)^2$. Here we use (3.19) with $\bar{\delta}_n = \log(n)^2/n$ to get that uniformly for $m \leq (\log n)^2$,

$$\mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n} e^{-\rho j_n - j_n^\beta} e^{-m-1}] \leq \frac{c}{\sqrt{n}} e^{-n\Lambda^*(\rho)} e^{-\Lambda(\alpha)j_n - \Lambda(\alpha)i_n - n\alpha\delta_n + \alpha\rho j_n + \alpha j_n^\beta + \alpha m}.$$

For $\mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m]$ we can use the bound (3.24) to get

$$\begin{aligned} & \sum_{0 \leq m \leq (\log n)^2} \mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n - \rho j_n - j_n^\beta} e^{-m-1}] \mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m] \\ & \leq \frac{c}{\sqrt{n}} \frac{e^{-n\Lambda^*(\rho)}}{1 - e^{-\varepsilon}} j_n^c \exp\{-n\alpha\delta_n - \Lambda(\alpha)i_n - \varepsilon j_n^\beta + c j_n \varepsilon^2\}. \end{aligned}$$

Putting both terms together yields

$$\begin{aligned} & \mathbb{P}[U_{n, n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} > e^{\rho j_n + j_n^\beta}] \\ & \leq c \left(\frac{1}{\sqrt{n}} + e^{-\varepsilon(\log n)^2} \right) \frac{e^{-n\Lambda^*(\rho)}}{1 - e^{-\varepsilon}} j_n^c \exp\{-n\alpha\delta_n - \Lambda(\alpha)i_n - \varepsilon j_n^\beta + c j_n \varepsilon^2\}. \end{aligned}$$

From this point, the desired bound (3.22) follows, if one takes $\varepsilon = (\log n)^{-1/2}$, since this choice gives

$$\begin{aligned} & \mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} > e^{\rho j_n + j_n^\beta}] \\ & \leq \frac{c}{\sqrt{n}} e^{-n\Lambda^*(\rho)} (\log n)^c \exp\{-n\alpha\delta_n - \Lambda(\alpha)i_n - \delta(\log n)^{\beta-\frac{1}{2}}\} = o\left(\frac{e^{-n\Lambda^*(\rho)-\alpha n\delta_n - \Lambda(\alpha)i_n}}{\sqrt{n}}\right). \end{aligned}$$

for sufficiently small δ .

STEP 2: TRUNCATED MOMENTS OF H_{j_n} . We have

$$\lim_{n \rightarrow \infty} \lambda(\alpha)^{-j_n} \mathbb{E}[H_{j_n}^\alpha \mathbf{1}_{\{H_{j_n} \leq e^{\rho j_n + j_n^\beta}\}}] = c_H,$$

where c_H is the value of the limit in (3.21). To make this evident, note that evoking (3.24) once again, we have

$$\begin{aligned} \mathbb{E}[H_{j_n}^\alpha; H_{j_n} > e^{\rho j_n + j_n^\beta}] & \leq \sum_{m \geq 0} \mathbb{E}[H_{j_n}^\alpha; e^{\rho j_n + j_n^\beta} e^m < H_{j_n} \leq e^{\rho j_n + j_n^\beta} e^{m+1}] \\ & \leq \sum_{m \geq 0} e^{\alpha\rho j_n + \alpha j_n^\beta} e^{\alpha(m+1)} \mathbb{P}[H_{j_n} > e^{\rho j_n + j_n^\beta} e^m] \\ & \leq \frac{1}{1 - e^{-\varepsilon}} c_j^c \exp\{-\varepsilon j_n^\beta + c_j n \varepsilon^2\} \lambda(\alpha)^{j_n}. \end{aligned}$$

If we put $\varepsilon = (\log n)^{-1/2}$, we get

$$\lim_{n \rightarrow \infty} \lambda(\alpha)^{-j_n} \mathbb{E}[H_{j_n}^\alpha \mathbf{1}_{\{H_{j_n} > e^{\rho j_n + j_n^\beta}\}}] = 0.$$

So the claim in this step follows.

STEP 3: CONCLUSION. In view of STEP 1 it is sufficient to justify

$$\mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} \leq e^{\rho j_n + j_n^\beta}] \sim c_H C_V \frac{e^{-n\Lambda^*(\rho)}}{\sqrt{n}} e^{-n\alpha\delta_n - \Lambda(\alpha)i_n} \quad \text{as } n \rightarrow \infty.$$

For this purpose, by (3.19) and the fact that H_{j_n} is integer valued, we can write

$$\begin{aligned} \mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n}, H_{j_n} \leq e^{\rho j_n + j_n^\beta}] & = \mathbb{P}[H_{j_n} \bar{V}_{j_n, n-i_n} > e^{n\rho+n\delta_n}, 1 \leq H_{j_n} \leq e^{\rho j_n + j_n^\beta}] \\ & = \sum_{k=1}^{\lfloor \exp\{\rho j_n + j_n^\beta\} \rfloor} \mathbb{P}[V_{n-j_n-i_n} > e^{n\rho+n\delta_n - \log k}] \mathbb{P}[H_{j_n} = k] \\ & = (1 + o(1)) C_V \frac{1}{\sqrt{n}} e^{-n\Lambda^*(\rho) - n\alpha\delta_n - \Lambda(\alpha)i_n} \cdot \lambda(\alpha)^{-j_n} \mathbb{E}[H_{j_n}^\alpha \mathbf{1}_{\{H_{j_n} \leq e^{\rho j_n + j_n^\beta}\}}]. \end{aligned}$$

For the last equality we use that the sequence $\tilde{\delta}_n = \frac{1}{1 \wedge \rho} n^{\frac{1}{2}} (\delta_n + \frac{\rho j_n + j_n^\beta}{n})$ converges to 0 and refer to (3.19). An appeal to the STEP 2 concludes the proof. \square

3.3. Large deviations of Z_n

Now we will prove that the sequence $\{Z_n\}_{n \geq 0}$ has the property (P_α) , which immediately entails Theorem 2.1.

Lemma 3.7. *Assume that conditions of Theorem 2.1 are satisfied. Then the sequence $\{Z_n\}_{n \geq 0}$ has the property (P_α) with constant $C_1(\rho)$.*

Proof. The first condition (3.18) follows from Lemma 3.1. To prove (3.19) we estimate deviations of Z_n from its quenched mean $\Pi_n = \mathbb{E}[Z_n | \mathcal{Q}]$. For this purpose we will use below (and in the consecutive lemmas) the following

equality valid for any $j \leq n$

$$(3.25) \quad Z_j \Pi_{j,n} = Z_n + \sum_{k=j+1}^n (A_{k-1} Z_{k-1} - Z_k) \Pi_{k,n}.$$

Take $j_n = K \lfloor \log n \rfloor$ for some large integer K that will be specified below. Let $U_{n,j} = Z_{j_n} \Pi_{j_n,j}$. Fix an arbitrary sequence $\{\bar{\delta}_n\}_{n \geq 0}$ of non-negative numbers converging to 0. Let $\{i_n\}_{n \geq 0}$ be any sequence of integers and let $\{\delta_n\}_{n \geq 0}$ be any sequence of real numbers such that condition (3.20) holds. Combining (3.25), Lemma 3.3 and the Chebyshev inequality, yields that for any positive t we have

$$\begin{aligned} \mathbb{P}[|Z_{n-i_n} - U_{n,n-i_n}| > t] &\leq \mathbb{P}\left[\left|\sum_{k=j_n+1}^{n-i_n} (A_{k-1} Z_{k-1} - Z_k) \Pi_{k,n-i_n}\right| > \sum_{k=j_n+1}^{n-i_n} \frac{t}{2k^2}\right] \\ &\leq \sum_{k=j_n+1}^{n-i_n} \mathbb{P}\left[|A_{k-1} Z_{k-1} - Z_k| \Pi_{k,n-i_n} > \frac{t}{2k^2}\right] \\ &\leq c \sum_{k=j_n+1}^{n-i_n} \gamma^k \lambda(\alpha)^{n-i_n-k} k^{2\alpha} t^{-\alpha} \\ &\leq c \lambda(\alpha)^{n-i_n} t^{-\alpha} \cdot \sum_{k=j_n+1}^{n-i_n} \left(\frac{\gamma}{\lambda(\alpha)}\right)^k k^{2\alpha} \leq c \eta^{j_n} j_n^{2\alpha} \lambda(\alpha)^{n-i_n} t^{-\alpha} \end{aligned}$$

for $\eta = \frac{\gamma}{\lambda(\alpha)} < 1$. Choosing $t = \delta e^{n\rho+n\delta_n}$ for some fixed $\delta > 0$ we obtain

$$(3.26) \quad \begin{aligned} \mathbb{P}[|Z_{n-i_n} - U_{n,n-i_n}| > \delta e^{n\rho+n\delta_n}] &\leq c \delta^{-\alpha} \eta^{j_n} (j_n)^{2\alpha} e^{-n\Lambda^*(\rho)} e^{-\alpha n \delta_n - \Lambda(\alpha) i_n} \\ &= \delta^{-\alpha} \cdot o(e^{-n\Lambda^*(\rho)} e^{-\alpha n \delta_n - \Lambda(\alpha) i_n} / \sqrt{n}) \end{aligned}$$

if only K is large enough. For an arbitrary small $\varepsilon > 0$ we have

$$\mathbb{P}[Z_{n-i_n} > e^{n\rho+n\delta_n}] \leq \mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n-\varepsilon}] + \mathbb{P}[|Z_{n-i_n} - U_{n,n-i_n}| > e^{n\rho+n\delta_n}(1 - e^{-\varepsilon})].$$

Combining (3.26) with Lemma 3.6 (its hypotheses are satisfied for $H_n = Z_n$ thanks to Lemmas 3.1 and 3.3) yields that for any sufficiently small $\varepsilon > 0$,

$$\limsup_{n \rightarrow \infty} \sqrt{n} e^{n\Lambda^*(\rho)} e^{\alpha n \delta_n + \Lambda(\alpha) i_n} \mathbb{P}[Z_{n-i_n} > e^{n\rho+n\delta_n}] \leq \frac{cZ e^{\alpha\varepsilon}}{\alpha\sigma(\alpha)\sqrt{2\pi}}.$$

Finally passing with $\varepsilon \rightarrow 0$ we obtain

$$\limsup_{n \rightarrow \infty} \sqrt{n} e^{n\Lambda^*(\rho)} e^{\alpha n \delta_n + \Lambda(\alpha) i_n} \mathbb{P}[Z_{n-i_n} > e^{n\rho+n\delta_n}] \leq \frac{cZ}{\alpha\sigma(\alpha)\sqrt{2\pi}}.$$

Similar arguments can be used to estimate the lower limit. Starting with the inequality

$$\mathbb{P}[Z_{n-i_n} > e^{n\rho+n\delta_n}] \geq \mathbb{P}[U_{n,n-i_n} > e^{n\rho+n\delta_n+\varepsilon}] - \mathbb{P}[|Z_{n-i_n} - U_{n,n-i_n}| > e^{n\rho+n\delta_n}(e^\varepsilon - 1)]$$

for any $\varepsilon > 0$, invoking (3.26), Lemma 3.6 and sending $\varepsilon \rightarrow 0$ we arrive at

$$\liminf_{n \rightarrow \infty} \sqrt{n} e^{n\Lambda^*(\rho)} e^{\alpha n \delta_n + \Lambda(\alpha) i_n} \mathbb{P}[Z_{n-i_n} > e^{n\rho+n\delta_n}] \geq \frac{cZ}{\alpha\sigma(\alpha)\sqrt{2\pi}}.$$

Thus we conclude the result. □

As a corollary from the above lemma we obtain:

Proof of Theorem 2.1. The main result follows from the property (P_α) of the sequence $\{Z_n\}$, choosing $\delta_n = j_n = 0$ in (3.19). □

4. Large deviations of T_t^Z

4.1. Preliminary bounds for T_t^Z

To estimate large deviations of T_t^Z it is convenient to consider

$$X_{n,j} = X_j = \begin{cases} Z_j & \text{for } j \leq n', \\ Z_{n'} \Pi_{n',j} & \text{for } j > n', \end{cases} \quad \text{and} \quad M_n = \max_{j \leq n} X_{n,j},$$

for $n' = n - \lfloor \theta \log n \rfloor$ and some fixed constant $\theta \in (0, \infty)$.

Lemma 4.1. *For any choice of $\theta \in (0, \infty)$, the array $\{X_{n,j}\}_{n \geq 0}$ has the property (P_α) with constant $C_1(\rho)$.*

Proof. Inequality (3.18) follows from Lemma 3.1 and the Markov inequality. Next, Lemma 3.6 applied for the pair $(H_n, V_n) = (\Pi_n, Z_n)$ and the array $U_{n,i} = X_i = Z_{n'} \Pi_{n',i}$ yields (3.19). \square

Lemma 4.2. *Suppose that the assumptions of Theorem 2.1 are in force and that n and t are related by (2.4). Then, for any fixed $\delta \in (0, 1)$ and arbitrary $K, \theta \in (0, \infty)$,*

$$\sup_{n^{-K} \leq a \leq n^K} a^\alpha \mathbb{P} \left[M_n > (1 + \delta)at, \max_{0 \leq j \leq n} Z_j \leq at \right] = o \left(\frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \right)$$

and

$$\sup_{n^{-K} \leq a \leq n^K} a^\alpha \mathbb{P} \left[M_n \leq (1 - \delta)at, \max_{0 \leq j \leq n} Z_j > at \right] = o \left(\frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \right).$$

Proof. The arguments for both claims are similar, therefore we prove here only the first part. Since $X_j = Z_j$ for $j \leq n'$, we have the following bound

$$\mathbb{P} \left[M_n > (1 + \delta)at, \max_{0 \leq j \leq n} Z_j \leq at \right] \leq \sum_{j=n'}^n \mathbb{P} [Z_j \leq at, X_j > (1 + \delta)at].$$

Combining (3.25), the Markov inequality and Lemma 3.3 (with $0 < \gamma < \lambda(\alpha)$) we obtain for $n' \leq j \leq n$,

$$\begin{aligned} \mathbb{P} [Z_j \leq at, X_j > (1 + \delta)at] &\leq \mathbb{P} \left[\sum_{k=n'+1}^j (A_{k-1} Z_{k-1} - Z_k) \Pi_{k,j} > \delta at \right] \\ (4.1) \qquad &\leq \sum_{k=n'+1}^j \mathbb{P} \left[|A_{k-1} Z_{k-1} - Z_k| \Pi_{k,j} > \frac{\delta at}{2k^2} \right] \\ &\leq \sum_{k=n'+1}^j c \gamma^k \lambda(\alpha)^{j-k} k^{2\alpha} (\delta at)^{-\alpha} \\ &\leq c \varepsilon^{n'} n^{2\alpha} \lambda(\alpha)^j (\delta at)^{-\alpha}, \end{aligned}$$

for some $\varepsilon \in (0, 1)$. If we sum over $n' \leq j \leq n$ and recall (3.23) we arrive at

$$\begin{aligned} \mathbb{P} \left[M_n > (1 + \delta)at, \max_{0 \leq j \leq n} Z_j \leq at \right] &\leq \sum_{j=n'}^n \mathbb{P} [Z_j \leq at, X_j > (1 + \delta)at] \\ &\leq \sum_{j=n'}^n c \varepsilon^{n'} n^{2\alpha} \lambda(\alpha)^j (\delta at)^{-\alpha} \end{aligned}$$

$$\leq c(\delta\alpha)^{-\alpha} \cdot \varepsilon^{n'} n^{2\alpha+1/2} \sum_{j=n'}^n \lambda(\alpha)^{j-n} \cdot \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}.$$

Since $\varepsilon^{n'} \cdot n^{2\alpha+1/2} \sum_{j=n'}^n \lambda(\alpha)^{j-n} = o(1)$ as $n \rightarrow \infty$, we conclude the proof. □

Lemma 4.3. *Assume that conditions of Theorem 2.1 are satisfied. There exist a parameter $0 < \beta < \alpha$ satisfying $\lambda(\beta) < \lambda(\alpha)$ and a constant $c(\alpha, \beta) \in (0, \infty)$ such that for any fixed integers L and N satisfying $L \geq 1$ and $-1 \leq N \leq L$, any sufficiently large t and sufficiently large $\theta > 0$, we have*

$$\sup_{n^{-K} \leq a, b \leq n^K} b^\beta a^{\alpha-\beta} \mathbb{P}[M_{n-L} > at, X_{n-N} > bt] \leq c(\alpha, \beta) \lambda(\alpha)^{-L} \lambda(\beta)^{L-N} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}$$

for any fixed $K > 0$.

Proof. First we explain how the parameter β is chosen. We take β to be any parameter such that $0 < \beta < \alpha$, $\lambda(\beta) < \lambda(\alpha)$, $\rho(\beta) > 0$ and additionally if $\alpha > 1$ we assume $\beta > 1$, $\lambda(\beta) > \lambda(1)$.

STEP 1. We will prove an auxiliary inequality. Let $k, N \in \mathbb{N}$ and $s > 0$ be arbitrary fixed constants and let $\{Z_{k,i}\}_{i \leq N}$ be independent, given \mathcal{Q} , copies of Z_k . Then there is a constant C such that

$$(4.2) \quad \mathbb{P} \left[\sum_{i=1}^N Z_{k,i} > s \right] \leq CN^\beta s^{-\beta} \lambda^k(\beta).$$

Assume first that $\beta \leq 1$. Applying the Markov inequality followed by the Jensen inequality

$$\mathbb{P} \left[\sum_{i=1}^N Z_{k,i} > s \right] \leq s^{-\beta} \mathbb{E} \left[\mathbb{E} \left[\left(\sum_{i=1}^N Z_{k,i} \right)^\beta \middle| \mathcal{Q} \right] \right] \leq s^{-\beta} \mathbb{E} \left[\mathbb{E} \left[\sum_{i=1}^N Z_{k,i} \middle| \mathcal{Q} \right]^\beta \right] \leq s^{-\beta} N^\beta \lambda(\beta)^k.$$

If $\beta > 1$, notice that thanks to the choice of the parameter hypothesis (H3) of Theorem 2.1 holds with α replaced by β and therefore we can invoke Lemma 3.1 to ensure

$$\mathbb{E} Z_k^\beta \leq C \lambda(\beta)^k.$$

Using the Markov inequality, conditional independence of $\{Z_{k,i}\}_{i \geq 1}$ and inequality (3.9), we obtain

$$\mathbb{P} \left[\sum_{i=1}^N Z_{k,i} > s \right] \leq s^{-\beta} \mathbb{E} \left[\mathbb{E} \left[\left(\sum_{i=1}^N Z_{k,i} \right)^\beta \middle| \mathcal{Q} \right] \right] \leq C_\beta s^{-\beta} N^{\frac{\beta}{2} \vee 1} \mathbb{E} [Z_k^\beta] \leq C_\beta C s^{-\beta} N^\beta \lambda(\beta)^k.$$

This completes proof of (4.2).

STEP 2. Denote $\delta = \frac{\lambda(\beta)}{\lambda(\alpha)} < 1$. We may write

$$\mathbb{P}[M_{n-L} > at, X_{n-N} > bt] \leq \sum_{j \leq n-L} \mathbb{P}[X_j > at, X_{n-N} > bt].$$

We will estimate the sum term by term. To do so, we will apply different bounds for small and large values of j .

STEP 2A. First we present a bound for $j \leq n'$, then $X_j = Z_j$ and we have

$$X_{n-N} = \sum_{i=1}^{Z_j} Z_{i,j,n'} \cdot \Pi_{n',n-N},$$

where $Z_{i,j,n'}$ is the number of progeny residing in the n' th generation of the i th individual alive at time j . Then, given \mathcal{Q} , $\{Z_{i,j,n'}\}_i$ are independent. Invoking inequality (3.18) for Z_j and (4.2) we have

$$\mathbb{P}[X_j > at, X_{n-N} > bt] = \sum_{m=0}^{\infty} \mathbb{P} \left[e^{m+1} at \geq Z_j > e^m at, \sum_{i=1}^{Z_j} Z_{i,j,n'} \cdot \Pi_{n',n-N} > bt \right]$$

$$\begin{aligned} &\leq \sum_{m=0}^{\infty} \mathbb{P}[Z_j > e^m at] \cdot \mathbb{P}\left[\sum_{i=1}^{\lceil e^{m+1} at \rceil} Z_{i,j,n'} \cdot \Pi_{n',n-N} > bt\right] \\ &\leq \sum_{m=0}^{\infty} c\lambda(\alpha)^j e^{-\alpha m} a^{-\alpha} t^{-\alpha} \lambda(\beta)^{n-N-j} b^{-\beta} a^\beta e^{\beta(m+1)} \\ &\leq c(\alpha, \beta) b^{-\beta} a^{\beta-\alpha} \delta^{n-j-L} t^{-\Lambda^*(\rho)/\rho} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N}. \end{aligned}$$

Summing now over $j \leq n'$, we arrive at

$$\begin{aligned} &\sum_{0 \leq j \leq n'} \mathbb{P}[X_j > at, X_{n-N} > bt] \\ (4.3) \quad &\leq c \sum_{0 \leq j \leq n'} b^{-\beta} a^{\beta-\alpha} \delta^{n-j-L} t^{-\Lambda^*(\rho)/\rho} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N} \\ &\leq cb^{-\beta} a^{\beta-\alpha} \delta^{\theta \log n - L} t^{-\Lambda^*(\rho)/\rho} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N} \leq cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\log t} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N}, \end{aligned}$$

where the last inequality holds provided $\theta > 0$ is sufficiently big.

STEP 2B. Now we give a bound for $j > n'$. In this case $X_j = Z_{n'} \Pi_{n',j}$. Take $B > 0$ to be any constant such that $-\alpha B + 1 + 2\beta K < 0$, whenever $\lambda(\alpha) \geq 1$ and $-\alpha B + 1 - \theta \Lambda(\alpha) + 2\beta K < 0$ for $\lambda(\alpha) < 1$. Consider the decomposition

$$\begin{aligned} \mathbb{P}[X_j > at, X_{n-N} > bt] &\leq \mathbb{P}[X_j > ate^{B \log n}, X_{n-N} > bt] \\ &\quad + \mathbb{P}[ate^{B \log n} \geq X_j > at, X_{n-N} > bt] = I_1 + I_2. \end{aligned}$$

To bound the first term invoke (3.18) for X_j and notice $1 \leq (a/b)n^{2K}$ (recall $a, b \in [n^{-K}, n^K]$):

$$\begin{aligned} I_1 &\leq \mathbb{P}[X_j \geq ate^{B \log n}] \leq c(at)^{-\alpha} n^{-\alpha B} \lambda(\alpha)^j \leq ct^{-\Lambda^*(\rho)/\rho} a^{-\alpha} n^{-\alpha B} \lambda(\alpha)^{j-n} \\ &\leq ct^{-\Lambda^*(\rho)/\rho} a^{\beta-\alpha} b^{-\beta} \frac{1}{\log t} n^{-\alpha B + 1 + 2\beta K} \lambda(\alpha)^{-n+j+L} \lambda(\alpha)^{-L} \leq ca^{\beta-\alpha} b^{-\beta} t^{-\Lambda^*(\rho)/\rho} \frac{1}{\log t} \lambda(\alpha)^{-L}. \end{aligned}$$

Turning our attention to the second term I_2 we apply Lemma 4.1 and (3.19), which gives the uniform estimates, combined with the same procedure as the one used in the previous step. Using the union bound

$$\begin{aligned} I_2 &= \mathbb{P}[ate^{B \log n} \geq X_j > at, X_{n-N} > bt] \\ &\leq \sum_{m=0}^{\lceil B \log n - 1 \rceil} \mathbb{P}[ate^{m+1} \geq X_j > ate^m, X_{n-N} > bt] \\ &\leq \sum_{m=0}^{\lceil B \log n - 1 \rceil} \mathbb{P}[X_j > ate^m] \mathbb{P}\left[\Pi_{j,n-N} > \frac{b}{a} e^{-m-1}\right] \\ &\leq \sum_{m=0}^{\lceil B \log n - 1 \rceil} c \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \lambda(\alpha)^{j-n} a^{-\alpha} e^{-\alpha m} b^{-\beta} a^\beta e^{\beta(m+1)} \lambda(\beta)^{n-j-N} \\ &\leq \sum_{m=0}^{\lceil B \log n - 1 \rceil} cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} e^{(\beta-\alpha)m} \delta^{n-j-L} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N} \\ &\leq cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \delta^{n-j-L} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N}. \end{aligned}$$

Now combine the bounds for I_1 and I_2 to get

$$\mathbb{P}[X_j > at, X_{n-N} > bt] \leq cb^{-\beta} a^{\beta-\alpha} t^{-\Lambda^*(\rho)/\rho} \frac{1}{\log t} \lambda(\alpha)^{-L}$$

$$+ cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \delta^{n-j-L} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N}.$$

Summing over $n - L \geq j > n'$ establishes an estimate sufficient for our needs

$$(4.4) \quad \sum_{n-L \geq j > n'} \mathbb{P}[X_j > at, X_{n-N} > bt] \leq cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \lambda(\alpha)^{-L} \cdot \left(\frac{\theta \log n}{\sqrt{\log t}} + \lambda(\beta)^{L-N} \right)$$

STEP 2C. Combining the claims of previous steps (4.3) with (4.4) we estimate

$$\begin{aligned} & \mathbb{P}[M_{n-L} > at, X_{n-N} > bt] \\ & \leq \sum_{0 \leq j \leq n'} \mathbb{P}[X_j > at, X_{n-N} > bt] + \sum_{n-L \geq j > n'} \mathbb{P}[X_j > at, X_{n-N} > bt] \\ & \leq cb^{-\beta} a^{\beta-\alpha} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \lambda(\alpha)^{-L} \lambda(\beta)^{L-N} \left(\frac{1}{\sqrt{\log t}} + \frac{\theta \log n}{\sqrt{\log t}} \lambda(\beta)^{N-L} + 1 \right). \end{aligned}$$

Since the last term can be bounded by a constant, we conclude the proof. □

4.2. Lower and upper estimates

We will focus our attention on establishing that first passage time for the array $\{X_{n,j}\}_{n,j}$ is of the correct order.

Proposition 4.4. *Assume that conditions of Theorem 2.1 are satisfied and that n and t are related via (2.4). For any constant K there is a positive constant c such that for sufficiently big $\theta > 0$ and t large enough one has*

$$(4.5) \quad \frac{1}{c} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \leq a^\alpha \mathbb{P}[M_{n-1} \leq at, X_n > at] \leq \frac{ct^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}$$

for any $a \in [n^{-K}, n^K]$.

Proof. Firstly, note that the upper bound in (4.5) follows by invoking Lemma 4.3 with $L = 1$ and $N = 0$. To establish the lower bound, we denote for non-negative integer L and any pair of positive reals $0 < \gamma < r < 1$,

$$\mathcal{A} = \mathcal{A}(r, \gamma, L) = \left\{ \max_{n-L \leq j \leq n-1} \Pi_{n-L,j} \leq r\gamma^{-1}, \Pi_{n-L,n} > \gamma^{-1} \right\}$$

and

$$\mathcal{B} = \mathcal{B}(r, \gamma, L) = \{M_{n-L-1} \leq at, \gamma at \leq X_{n-L-1} < \gamma r^{-1} at\}.$$

Then, by a direct calculation, it can be easily verified that

$$\mathcal{A} \cap \mathcal{B} \subseteq \{M_{n-1} \leq at, X_n > at\}.$$

By independence of \mathcal{A} and \mathcal{B} we are allowed to treat probabilities of respective events separately. To bound $\mathbb{P}[\mathcal{B}]$ note that

$$\mathbb{P}[\mathcal{B}] = \mathbb{P}[\gamma at \leq X_{n-L-1} < \gamma r^{-1} at] - \mathbb{P}[M_{n-L-1} > at, \gamma at \leq X_{n-L-1} < \gamma r^{-1} at].$$

The first probability, by (3.19) exhibits the following asymptotic behaviour

$$\mathbb{P}[\gamma at \leq X_{n-L-1} < \gamma r^{-1} at] \sim c_1(\alpha, r) \gamma^{-\alpha} \lambda(\alpha)^{-L} \frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \lambda(\alpha)^{-\Theta(t)}$$

while asymptotic of the second probability can be bounded by another appeal to Lemma 4.3 with $N = L$ by

$$\begin{aligned} \mathbb{P}[M_{n-L-1} > at, \gamma at \leq X_{n-L} < \gamma r^{-1} at] & \leq \mathbb{P}[M_{n-L-1} > at, X_{n-L} \geq \gamma at] \\ & \leq c_2(\alpha, \beta) \gamma^{-\beta} \lambda(\alpha)^{-L} \frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}. \end{aligned}$$

If we put everything together, and take $\delta > 0$ such that $\lambda(\alpha)^{-\Theta(t)} > \delta$, we will arrive at the conclusion that, uniformly in $a \in [n^{-K}, n^K]$,

$$\mathbb{P}[M_{n-1} \leq at, X_n > at] \geq \mathbb{P}[\mathcal{A}]a^{-\alpha}(\delta c_1(\alpha, r)\gamma^{-\alpha} - c_2(\alpha, \beta)\gamma^{-\beta} + o(1))\lambda(\alpha)^{-L} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}.$$

We need to ensure, that for a proper choice of γ, r and L ,

$$\mathbb{P}[\mathcal{A}](\delta c_1(\alpha, r)\gamma^{-\alpha} - c_2(\alpha, \beta)\gamma^{-\beta}) > 0.$$

To do so, first take r such that $\mathbb{P}[A > r^{-2}] > 0$ (such r exists since $\mathbb{P}[A > 1] > 0$) and then take γ sufficiently small such that

$$\delta c_1(\alpha, r)\gamma^{-\alpha} - c_2(\alpha, \beta)\gamma^{-\beta} > 0 \quad \text{and} \quad \gamma < r^2.$$

Finally choose L such that (changing γ if necessary) $\mathbb{P}[r^2\gamma^{-1} < A^{L-1} < r\gamma^{-1}] > 0$. The constants chosen in this way allow us to write, since $r^2\gamma^{-1} > 1$,

$$\begin{aligned} \mathbb{P}[\mathcal{A}] &= \mathbb{P}\left[\max_{1 \leq j \leq L-1} \Pi_j \leq r\gamma^{-1}, \Pi_L > \gamma^{-1}\right] \\ &\geq \mathbb{P}[\Pi_{L-1} \geq \Pi_{L-2} \geq \dots \geq \Pi_1, r^2\gamma^{-1} < \Pi_L < r\gamma^{-1}, A_{L-1} > r^{-2}] \\ &\geq \mathbb{P}[A_{L-1} > r^{-2}] \prod_{i=0}^{L-2} \mathbb{P}\left[r^{\frac{2}{L-1}}\gamma^{-\frac{1}{L-1}} < A_i < r^{\frac{1}{L-1}}\gamma^{-\frac{1}{L-1}}\right] \\ &> 0. \end{aligned}$$

□

4.3. Conclusions

Lemma 4.5. *Suppose that conditions of Theorem 2.1 are satisfied and let n and t be related via (2.4). For any $K > 0$ and sufficiently large $\theta > 0$ we have*

$$\lim_{t \rightarrow \infty} a^\alpha \lambda(\alpha)^{-\Theta(t)} t^{\Lambda^*(\rho)/\rho} \sqrt{\log t} \mathbb{P}[M_{n-1} \leq at, X_n > at] = \mathcal{C}_4(\rho) \in (0, \infty)$$

uniformly for $a \in [n^{-K}, n^K]$.

Proof. Take large L and write

$$\begin{aligned} &\mathbb{P}\left[X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L, j} \leq at, X_n > at\right] \\ &= \mathbb{P}\left[X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L, j} \leq at, X_n > at, \max_{j < n-L} X_j > at\right] + \mathbb{P}[M_{n-1} \leq at, X_n > at]. \end{aligned}$$

Using Lemma 4.3 we infer that the first term on the right-hand side has arbitrarily small contribution since it can be bounded uniformly with respect to $a \in [n^{-K}, n^K]$ via

$$\begin{aligned} \mathbb{P}\left[X_n > at, \max_{j < n-L} X_j > at\right] &= \mathbb{P}[M_{n-L} > at, X_n > at] \\ &\leq c(\alpha, \beta)\lambda(\alpha)^{-L}\lambda(\beta)^L \frac{a^{-\alpha}t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} = c(\alpha, \beta)\delta^L \frac{a^{-\alpha}t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}, \end{aligned}$$

where $\beta < \alpha$ and $\delta = \frac{\lambda(\beta)}{\lambda(\alpha)} < 1$. Choosing large L , δ^L can be arbitrary small. By Proposition 4.4 in order to conclude the lemma it will be sufficient to show that

$$a^\alpha \lambda(\alpha)^{-\Theta(t)} t^{\Lambda^*(\rho)/\rho} \sqrt{\log t} \mathbb{P}\left[X_{n-L-1} \max_{n-L \leq j < n} \Pi_{n-L, j} \leq at, X_n > at\right]$$

converges for L arbitrarily large. Note that the probability in question can be decomposed in the following fashion

$$\begin{aligned} & \mathbb{P}\left[X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L,j} \leq at, X_n > at\right] \\ &= \mathbb{P}\left[X_{n-L} \leq ate^{-(\log t)^{1/4}}, X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L,j} \leq at, X_n > at\right] \\ & \quad + \mathbb{P}\left[ate^{-(\log t)^{1/4}} < X_{n-L} \leq at, X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L,j} \leq at, X_n > at\right] \\ &= J_1 + J_2. \end{aligned}$$

Consider the first term for the moment. Our aim is to prove that its contribution is negligible. We will utilize the same procedure as the one used in the proof of Lemma 4.3. For $\beta > \alpha$, in view of (3.18), we have

$$\begin{aligned} J_1 &\leq \mathbb{P}\left[X_{n-L} \leq ate^{-(\log t)^{1/4}}, X_n > at\right] \\ &= \sum_{m \geq 0} \mathbb{P}\left[e^{-m-1} ate^{-(\log t)^{1/4}} < X_{n-L} \leq e^{-m} ate^{-(\log t)^{1/4}}, X_n > at\right] \\ &\leq \sum_{m \geq 0} \mathbb{P}\left[e^{-m-1} ate^{-(\log t)^{1/4}} < X_{n-L}\right] \mathbb{P}\left[\Pi_{n-L,n} > e^m e^{(\log t)^{1/4}}\right] \\ &\leq \sum_{m \geq 0} c\lambda(\alpha)^{n-L} a^{-\alpha} t^{-\alpha} e^{\alpha(\log t)^{1/4}} e^{\alpha(m+1)} \lambda(\beta)^{L+1} e^{-\beta(\log t)^{1/4}} e^{-\beta m} \\ &= ca^{-\alpha} t^{-\Lambda^*(\rho)/\rho} e^{(\alpha-\beta)(\log t)^{1/4}} \left(\frac{\lambda(\beta)}{\lambda(\alpha)}\right)^L \sum_{m \geq 0} e^{\alpha(m+1)} e^{-\beta m} = o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right), \end{aligned}$$

for some $c = c(\alpha, \beta, L)$. Left with an investigation of J_2 we note that by the same arguments as above one can deduce that

$$\mathbb{P}\left[X_{n-L} > ate^{-(\log t)^{1/4}}, \Pi_{n-L,n} > e^{(\log t)^{1/4}}\right] = o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right)$$

and as a consequence

$$\begin{aligned} J_2 &= \mathbb{P}\left[ate^{-(\log t)^{1/4}} < X_{n-L} \leq at, X_{n-L} \max_{n-L \leq j < n} \Pi_{n-L,j} \leq at, \right. \\ & \quad \left. X_n > at, \Pi_{n-L,n} \leq e^{(\log t)^{1/4}}\right] + o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right). \end{aligned}$$

By conditioning on $M'_{L-1} = \max_{n-L+1 \leq j < n} \Pi_{n-L+1,j}$ and $\Pi'_L = \Pi_{n-L+1,n}$ we have

$$\begin{aligned} J_2 &= \int_{0 \leq y \leq x < e^{(\log t)^{1/4}}} \mathbb{P}\left[atx^{-1} < X_{n-L} < at(1 \wedge y^{-1})\right] \mathbb{P}\left[M'_{L-1} \in dy, \Pi'_L \in dx\right] \\ & \quad + o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right). \end{aligned}$$

Now, refer again to the (P_α) property of $\{X_{n,j}\}_{n,j}$ and apply (3.19) with $j_n = \lfloor L + \theta \log n \rfloor$, $\bar{\delta}_n = Cn^{-\frac{1}{4}}$ and $\delta_n = \frac{\log(ay^{-1})}{n}$, where $C > 0$ is such that $|\sqrt{n}\delta_n| = n^{-1/2} \log(y/a) \leq n^{-1/4}C$. We infer that

$$\mathbb{P}\left[X_{n-L} \geq aty^{-1}\right] = c(\alpha) a^{-\alpha} \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} y^\alpha e^{-L\Lambda(\alpha)} (1 + o(1)).$$

We conclude that

$$J_2 = C(\alpha) a^{-\alpha} \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} e^{-L\Lambda(\alpha)} \mathbb{E}\left[\left((\Pi'_L)^\alpha - (M'_{L-1})^\alpha\right)_+\right] (1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad \square$$

Proof of Theorem 2.2. We focus on the proof of precise pointwise estimates (2.7) (Step 1), since it implies almost immediately both (2.5) and (2.6) (Step 2 and Step 3). Let us mention that both limits (2.5) and (2.6) can be proved in a much simpler way, e.g. using similar techniques to those presented in Lalley [28] and in particular omitting the tedious proofs of Lemmas 4.3 and 4.5.

STEP 1. First we prove that for any fixed constant K we have

$$(4.6) \quad \mathbb{P}[T_{at}^Z = n] = \mathbb{P}\left[\max_{j \leq n-1} Z_j \leq at, Z_n > at\right] \sim C_4(\rho) a^{-\alpha} \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}},$$

uniformly for $a \in [n^{-K}, n^K]$. Observe that formula (4.6) implies (2.7). Indeed for n, t, Θ as in (2.4),

$$\mathbb{P}[T_t^Z = n] = \mathbb{P}\left[\max_{j \leq n-1} Z_j \leq t, Z_n > t\right] \sim C_4(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}.$$

Note that (4.6) is indeed much stronger than (2.7), because the estimates are uniform, however uniformity will be needed to deduce (2.5) and (2.6).

Recall that $n' = \lfloor n - \theta \log n \rfloor$. In view of all previous considerations, we are left with approximation of Z_n with $X_n = Z_{n'} \Pi_{n',n}$ as $n \rightarrow \infty$.

STEP 1A. We prove upper estimate

$$(4.7) \quad \limsup_{t \rightarrow \infty} a^\alpha \lambda(\alpha)^{\Theta(t)} t^{\Lambda^*(\rho)/\rho} \sqrt{\log t} \mathbb{P}[T_{at}^Z = n] \leq C_4(\rho).$$

For this purpose we fix $\delta > 0$ and write

$$\begin{aligned} \mathbb{P}\left[\max_{j \leq n-1} Z_j \leq at, Z_n > at\right] &\leq \mathbb{P}\left[\max_{j \leq n-1} Z_j \leq at, M_{n-1} > (1 + \delta)at\right] + \mathbb{P}[X_n \leq (1 - \delta)at, Z_n > at] \\ &\quad + \mathbb{P}[(1 - \delta)at < X_n \leq (1 + \delta)at] + \mathbb{P}[M_{n-1} \leq (1 + \delta)at, X_n > (1 + \delta)at] \\ &= I + II + III + IV. \end{aligned}$$

In view of Lemma 4.5 the last expression has the required asymptotic behaviour, that is

$$(4.8) \quad \lim_{t \rightarrow \infty} \sqrt{\log(t)} a^\alpha \lambda(\alpha)^{\Theta(t)} t^{\Lambda^*(\rho)/\rho} \cdot IV = (1 + \delta)^{-\alpha} C_4(\rho).$$

Thus we need to prove that the other terms are negligible. The first one, namely I, is of the order

$$(4.9) \quad I = o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right)$$

by the merit of Lemma 4.2. Arguing as in the proof of Lemma 4.2 (see (4.1)), one can deduce

$$(4.10) \quad II = \mathbb{P}[Z_n > at, X_n \leq (1 - \delta)at] = o\left(\frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right).$$

By an appeal to Lemma 3.6 we estimate III via

$$(4.11) \quad \mathbb{P}[(1 - \delta)at < X_n \leq (1 + \delta)at] \leq h(\delta) \frac{a^{-\alpha} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}},$$

where $h(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Combining (4.8), (4.9), (4.10) with (4.11) and then passing with δ to 0 we conclude (4.7).

STEP 1B. To get the lower bound, apply the same procedure. However in this case, for fixed $\delta > 0$ we use the following inequality

$$\begin{aligned} &\mathbb{P}\left[\max_{j \leq n-1} Z_j \leq at, Z_n > at\right] \\ &\geq \mathbb{P}[M_{n-1} \leq (1 - \delta)at, X_n > (1 - \delta)at] \\ &\quad - \mathbb{P}\left[\max_{j \leq n-1} Z_j > at, M_{n-1} \leq (1 - \delta)at\right] - \mathbb{P}[(1 - \delta)at < X_n \leq (1 + \delta)at] - \mathbb{P}[X_n > (1 + \delta)at, Z_n \leq at] \end{aligned}$$

The same arguments as in the Step 1a (the first term on the right side above dominates, whereas all the remaining are negligible) lead us to the lower bound

$$\liminf_{t \rightarrow \infty} a^\alpha \lambda(\alpha)^{\Theta(t)} t^{\Lambda^*(\rho)/\rho} \sqrt{\log t} \mathbb{P}[T_{at}^Z = n] \geq C_4(\rho),$$

which together with (4.7) completes proof of the first step.

STEP 2. Now we prove (2.5). Note, that we only consider $\rho > \rho_0$, so in particular $\lambda(\alpha) > 1$. We will use below the following bound

$$\mathbb{P}\left[\max_{0 \leq j \leq n'} Z_j > t\right] \leq \mathbb{P}[W_{n'} > t] = o\left(\frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right),$$

for $n' = n - \lceil \theta \log n \rceil$ with for sufficiently large $\theta > 0$. We postpone its proof to the next section (see Corollary 5.6). Therefore, by (4.6), with n replaced by $n - j$ and $K = \rho(\theta + 1)$, we obtain

$$\begin{aligned} \mathbb{P}[T_t^Z \leq n] &\sim \mathbb{P}[\lfloor n - \theta \log n \rfloor \leq T_t^Z \leq n] \\ &= \sum_{j=0}^{\lceil \theta \log n \rceil} \mathbb{P}\left[\max_{i < n-j} Z_i \leq t \text{ and } Z_{n-j} > t\right] \\ &= \sum_{j=0}^{\lceil \theta \log n \rceil} \mathbb{P}\left[\max_{i < n-j} Z_i \leq e^{\rho(n-j)} e^{\rho j} e^{\rho \Theta(t)} \text{ and } Z_{n-j} > e^{\rho(n-j)} e^{\rho j} e^{\rho \Theta(t)}\right] \\ &\sim C_4(\rho) \sum_{j=0}^{\lceil \theta \log n \rceil} \frac{e^{-\alpha \rho j} e^{-\alpha \rho \Theta(t)} e^{-\Lambda^*(\rho)(n-j)}}{\sqrt{\rho(n-j)}} \\ &\sim C_4(\rho) \lambda(\alpha)^{-\Theta(t)} \frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \sum_{j=0}^{\lceil \theta \log n \rceil} \lambda(\alpha)^{-j} \\ &\sim \frac{C_4(\rho) \lambda(\alpha)^{1-\Theta(t)} t^{-\Lambda^*(\rho)/\rho}}{\lambda(\alpha) - 1} \frac{1}{\sqrt{\log t}}, \end{aligned}$$

which completes the proof of (2.5).

STEP 3. To prove (2.6) we proceed as above. This time $\rho < \rho_0$ and $\lambda(\alpha) < 1$. Put $\bar{n} = n + \lfloor D \log n \rfloor$ for some large constant D . We begin by estimating with the help of Markov inequality and Lemma 3.1,

$$\mathbb{P}\left[\max_{j > \bar{n}} Z_j > t\right] \leq \sum_{j=\bar{n}}^{\infty} \mathbb{P}[Z_j > t] \leq C t^{-\alpha} \lambda(\alpha)^{\bar{n}} = o\left(\frac{t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}}\right)$$

provided that D is large enough. Next, applying (4.6)

$$\begin{aligned} \mathbb{P}[T_t^Z \geq n] &\sim \mathbb{P}[n \leq T_t^Z \leq \lfloor n + D \log n \rfloor] \\ &= \sum_{j=0}^{\lfloor D \log n \rfloor} \mathbb{P}\left[\max_{i < n+j} Z_i \leq t \text{ and } Z_{n+j} > t\right] \\ &= \sum_{j=0}^{\lfloor D \log n \rfloor} \mathbb{P}\left[\max_{i < n+j} Z_i \leq e^{\rho(n+j)} e^{-\rho j} e^{\rho \Theta(t)} \text{ and } Z_{n+j} > e^{\rho(n+j)} e^{-\rho j} e^{\rho \Theta(t)}\right] \\ &\sim C_4(\rho) \sum_{j=0}^{\lfloor D \log n \rfloor} \frac{e^{\alpha \rho j} e^{-\alpha \rho \Theta(t)} e^{-(n+j)\Lambda^*(\rho)}}{\sqrt{\rho(n+j)}} \\ &\sim C_4(\rho) \frac{\lambda(\alpha)^{-\Theta(t)} t^{-\Lambda^*(\rho)/\rho}}{\sqrt{\log t}} \sum_{j=0}^{\lfloor D \log n \rfloor} \lambda(\alpha)^j \end{aligned}$$

$$\sim \frac{\mathcal{C}_4(\rho)\lambda(\alpha)^{-\Theta(t)} t^{-\Lambda^*(\rho)/\rho}}{1 - \lambda(\alpha) \sqrt{\log t}}.$$

We conclude the proof. □

5. Estimates for W

5.1. Preliminary bounds for the total population W_n and the perpetuity R_n

In this section we consider the total population up to generation n of BPPE Z , $W_n = \sum_{k=0}^n Z_k$. Recall that now we assume the process Z is in the subcritical regime, hence it dies out with probability 1.

The arguments leading to large deviation estimates for W follow the same idea as the one for Z , namely approximation of W_n by its conditional mean. As we already now, the quenched expectation of $R_n = \mathbb{E}[W_n | \mathcal{Q}]$ forms a perpetuity sequence (see (2.8)). In order to be able to execute the procedure, denote for $m < n \leq \infty$,

$$W_{m,n} = W_n - W_m = \sum_{k=m+1}^n Z_k,$$

$$R_{m,n} = \sum_{k=m+1}^n \Pi_{m,k} = \sum_{k=m+1}^n \prod_{j=m}^{k-1} A_j = A_m + A_m A_{m+1} + \dots + A_m A_{m+1} \dots A_{n-1}.$$

Then

$$\mathbb{E}[W_{m,n} | Z_m, \mathcal{Q}] = Z_m R_{m,n}.$$

Notice that since $\mathbb{E} \log A < 0$, $R_{m,\infty}$ is finite a.s. for every m as a convergent series. In particular $W_{m,\infty}$ is also finite a.s. for every m .

Lemma 5.1. *For $n \in \mathbb{N} \cup \{\infty\}$ and $n > m$ the following formula holds*

$$W_{m,n} - Z_m R_{m,n} = \sum_{k=m+1}^n (Z_k - A_{k-1} Z_{k-1})(1 + R_{k,n}).$$

Proof. Recall

$$(5.1) \quad Z_j - \Pi_j = \sum_{k=1}^j (Z_k - A_{k-1} Z_{k-1}) \Pi_{k,j}.$$

Similarly, taking a shorter telescopic sum yields, for $m < j$,

$$Z_j - Z_m \Pi_{m,j} = \sum_{k=m+1}^j (Z_k - A_{k-1} Z_{k-1}) \Pi_{k,j}.$$

If we sum the expression above over $m + 1 \leq j \leq n$ and change the order of summation on the right-hand side we will arrive at

$$\begin{aligned} W_{m,n} - Z_m R_{m,n} &= \sum_{j=m+1}^n (Z_j - Z_m \Pi_{m,j}) = \sum_{j=m+1}^n \sum_{k=m+1}^j (Z_k - A_{k-1} Z_{k-1}) \Pi_{k,j} \\ &= \sum_{k=m+1}^n (Z_k - A_{k-1} Z_{k-1}) \sum_{j=k}^n \Pi_{k,j} = \sum_{k=m+1}^n (Z_k - A_{k-1} Z_{k-1})(1 + R_{k,n}), \end{aligned}$$

which proves the lemma. □

Now we present the proof of formula (1.6). This relation, under slightly different hypotheses, was proved by Afanasyev [1]. In our setting it follows from the calculations contained Kesten et al. [25], however it is not explicitly stated there. For the reader convenience, we present here a short argument.

Lemma 5.2. *Assume that the hypotheses of Theorem 2.1 are satisfied. If additionally (1.3) holds then*

$$\mathbb{P}[W_{0,\infty} > t] = \mathbb{P}\left[\sum_{n=0}^{\infty} Z_n > t\right] \sim c_Z(\alpha_0)C_{KG}t^{-\alpha_0},$$

where the constant $c_Z(\alpha_0)$ was defined in Lemma 3.1 and C_{KG} is the Kesten–Goldie constant (see (5.3) below).

Proof. To prove the claim we invoke Lemma 5.1 with $n = \infty$ yielding

$$(5.2) \quad W_{m,\infty} - Z_m R_{m,\infty} = \sum_{k=m+1}^{\infty} (Z_k - A_{k-1}Z_{k-1})(1 + R_{k,\infty})$$

and the Kesten–Goldie theorem (Kesten [24], Goldie [18]) who described the tail behavior of the perpetuity:

$$(5.3) \quad \lim_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[1 + R_{0,\infty} > t] = C_{KG} > 0.$$

In particular for some constant \tilde{C}_{KG} ,

$$\sup_{t \geq 0} t^{\alpha_0} \mathbb{P}[1 + R_{0,\infty} > t] \leq \tilde{C}_{KG}.$$

Observe that since $R_{m,\infty} \stackrel{d}{=} R_{0,\infty}$ in the above formula $R_{0,\infty}$ can be replaced by $R_{m,\infty}$ for any $m \in \mathbb{N}$. Let us fix an arbitrary $\varepsilon > 0$. Take $\gamma < \lambda(\alpha_0) = 1$ as in Lemma 3.1 and choose a constant M satisfying

$$(5.4) \quad \gamma^M \cdot 2^{1+\alpha_0} \tilde{C}_{KG} \sum_{k=1}^{\infty} k^{2\alpha_0} \gamma^k < \varepsilon^{1+\alpha_0} \quad \text{and} \quad (1 - \varepsilon)c_Z(\alpha_0) < \mathbb{E}Z_M^{\alpha_0} < (1 + \varepsilon)c_Z(\alpha_0).$$

We can write

$$(5.5) \quad \mathbb{P}[W_{0,\infty} > t] \leq \mathbb{P}[W_M \geq \varepsilon t] + \mathbb{P}[|W_{M,\infty} - Z_M R_{M,\infty}| > \varepsilon t] + \mathbb{P}[Z_M R_{M,\infty} > (1 - 2\varepsilon)t].$$

The first term can be bounded by taking $\delta \in (0, \alpha_\infty - \alpha_0)$ and writing with the help of Lemma 3.1 for some constant $c = c(\delta, M)$,

$$\mathbb{P}[W_M \geq \varepsilon t] \leq \sum_{k=1}^M \mathbb{P}\left[Z_k \geq \frac{\varepsilon t}{M}\right] \leq t^{-\alpha_0 - \delta} \varepsilon^{-\alpha_0 - \delta} M^{\alpha_0 + \delta} \sum_{k=1}^M c k^c \lambda(\alpha_0 + \delta)^k = o(t^{-\alpha_0})$$

as $t \rightarrow \infty$ while δ, M and ε remain fixed. Turning our attention to the second term we combine (5.2), (5.3) with Lemma 3.1, there is t_0 such that for $t > t_0$ we have

$$\begin{aligned} \mathbb{P}[|W_{M,\infty} - Z_M R_{M,\infty}| > \varepsilon t] &\leq \mathbb{P}\left[\sum_{k=M+1}^{\infty} |Z_k - A_{k-1}Z_{k-1}|(1 + R_{k,\infty}) > \varepsilon t\right] \\ &\leq \sum_{k=M+1}^{\infty} \mathbb{P}\left[|Z_k - A_{k-1}Z_{k-1}|(1 + R_{k,\infty}) > \frac{\varepsilon t}{2(k - M)^2}\right] \\ &\leq 2^{1+\alpha_0} \tilde{C}_{KG} \varepsilon^{-\alpha_0} t^{-\alpha_0} \sum_{k=M+1}^{\infty} (k - M)^{2\alpha_0} \gamma^k \leq \varepsilon t^{-\alpha_0}. \end{aligned}$$

Multiplying both sides of inequality (5.5) by t^{α_0} and passing to the upper limit, in view of (5.4) and the above estimates, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[W_{0,\infty} > t] \\ & \leq \limsup_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[|W_{M,\infty} - Z_M R_{M,\infty}| > \varepsilon t] + \limsup_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[Z_M R_{M,\infty} > (1 - \varepsilon)t] \\ & \leq \varepsilon + (1 - \varepsilon)^{-\alpha_0} (1 + \varepsilon) c_Z(\alpha_0) C_{KG}. \end{aligned}$$

Sending ε to 0 we have

$$\limsup_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[W_{0,\infty} > t] \leq c_Z(\alpha_0) C_{KG}.$$

For the corresponding lower bound we use the inequality

$$\mathbb{P}[W_{0,\infty} > t] \geq \mathbb{P}[Z_M R_{M,\infty} > (1 + 2\varepsilon)t] - \mathbb{P}[W_M \geq \varepsilon t] - \mathbb{P}[|W_{M,\infty} - Z_M R_{M,\infty}| > \varepsilon t].$$

Calculations similar to the ones used for the upper bound lead to

$$\liminf_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[W_{0,\infty} > t] \geq c_Z(\alpha_0) C_{KG},$$

and thus we conclude the proof. □

Now we state some elementary properties of W_n and R_n . The following lemma was proved in [11] (see proof of Theorem 2.1).

Lemma 5.3. *If $\alpha > \alpha_0$, then there exists a finite positive limit*

$$c_R(\alpha) = \lim_{n \rightarrow \infty} \lambda(\alpha)^{-n} \mathbb{E}[R_n^\alpha].$$

The next lemma was stated in [11] as Lemma 3.1. We prove below its counterpart in terms of W .

Lemma 5.4. *If $\alpha \geq \alpha_0$ and $\varepsilon < \min\{1/2, \alpha_\infty - \alpha\}$ and $\rho = \rho(\alpha)$, then one can find a constant $c = c(\alpha, \varepsilon) > 0$ such that for all N and M ,*

$$\mathbb{P}[R_N > e^M] \leq c N^{2(\alpha+1)} \exp\{-M(\alpha + \varepsilon) + N\Lambda(\alpha) + N\rho\varepsilon + cN\varepsilon^2\}.$$

Lemma 5.5. *If $\alpha \in [\alpha_0, \alpha_\infty)$ and the assumptions of Theorem 2.1 are in force. Then there is $\varepsilon > 0$ such that $\alpha + \varepsilon < \alpha_\infty$ and $\mathbb{E}[Z_1^{\alpha+\varepsilon}] < \infty$ one can find a constant $c = c(\alpha, \varepsilon) > 0$ such that for all N, M*

$$\mathbb{P}[W_N > e^M] \leq c N^{2(\alpha+c+1)} \exp\{-M(\alpha + \varepsilon) + N\Lambda(\alpha) + N\rho\varepsilon + \sigma(\alpha)N\varepsilon^2\}.$$

Proof. Invoking Lemma 3.3 with $\varepsilon < \delta$, we have

$$\begin{aligned} \mathbb{P}[W_N > e^M] & \leq \sum_{k=1}^N \mathbb{P}[Z_k > e^M (2k^2)^{-1}] \leq \sum_{k=1}^N \mathbb{E}[Z_k^{\alpha+\varepsilon}] e^{-M(\alpha+\varepsilon)} 2^{\alpha+\varepsilon} k^{2(\alpha+\varepsilon)} \\ & \leq c e^{-M(\alpha+\varepsilon)} \sum_{k=1}^N e^{\Lambda(\alpha+\varepsilon)k} k^{2(\alpha+c+\varepsilon)}. \end{aligned}$$

Since $\Lambda(\alpha + \varepsilon) \leq \Lambda(\alpha) + \rho(\alpha)\varepsilon + \sigma(\alpha)\varepsilon^2$, for sufficiently small ε ,

$$\begin{aligned} \mathbb{P}[W_N > e^M] & \leq c e^{-M(\alpha+\varepsilon)} \sum_{k=1}^N e^{(\Lambda(\alpha)+\rho(\alpha)\varepsilon+\varepsilon^2\sigma(\alpha))k} k^{2(\alpha+c+\varepsilon)} \\ & \leq c e^{-M(\alpha+\varepsilon)} e^{(\Lambda(\alpha)+\rho(\alpha)\varepsilon+\varepsilon^2\sigma(\alpha))N} N^{2(\alpha+c+\varepsilon)+1}. \end{aligned}$$

□

Corollary 5.6. Assume that $n = \lfloor \rho^{-1} \log t \rfloor$ for some $\rho(\alpha_0) < \rho < \rho(\alpha_\infty)$. Let α be chosen such that $\rho(\alpha) = \rho$. For $n' = \lfloor n - \theta \log n \rfloor$ with $\theta > 0$ sufficiently large we have

$$\mathbb{P}[W_{n'} > t] = o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho}).$$

In particular

$$\mathbb{P}\left[\max_{0 \leq j \leq n'} Z_j > t\right] = o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho}).$$

Proof. If we invoke Lemma 5.5 with $N = n'$, $M = \log t$ and $\varepsilon = (\log t)^{-1/2}$ we infer, that for some constant c (which is bounded by Lemma 3.1)

$$\begin{aligned} \mathbb{P}[W_{n'} > t] &\leq ct^{-\alpha-\varepsilon} (n')^{2(\alpha+1)} \exp\{n' \Lambda(\alpha) + n' \rho \varepsilon + \sigma(\alpha) n' \varepsilon^2\} \\ &\leq ct^{-\alpha-\varepsilon} n^{2(\alpha+\sigma(\alpha)+1)} \exp\{(n - \theta \log n) \Lambda(\alpha) + (n - \theta \log n) \rho \varepsilon\} \\ &\leq ct^{-\Lambda^*(\rho)/\rho} n^{2(\alpha+\sigma(\alpha)+1)} \exp\{-\theta \log n \Lambda(\alpha) - \theta \log n \rho \varepsilon\} \\ &\leq ct^{-\Lambda^*(\rho)/\rho} (\log t)^{2(\alpha+\sigma(\alpha)+1)-\theta \Lambda(\alpha)}. \end{aligned}$$

By choosing $\theta > 0$ sufficiently large we can make the last expression $o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho})$ and conclude the proof. \square

In view of Corollary 5.6 to determine the asymptotic of $\mathbb{P}[W_n > t]$ it suffices to investigate $\mathbb{P}[W_{n',n} > t]$. For the moment, we will focus our attention on showing how to approximate $W_{n',n}$ by its conditional mean.

Lemma 5.7. Under the assumptions of Theorem 2.1 fix $\rho = \rho(\alpha)$ for some $\alpha \in (\alpha_0, \alpha_\infty)$ and assume additionally (1.3) and that n and t are related via (2.4). We have

$$\mathbb{P}[|W_{n',n} - Z_{\tilde{n}} \Pi_{\tilde{n},n'} R_{n',n}| > t] = o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho}),$$

where $n' = \lceil n - \theta \log n \rceil$ and $\tilde{n} = \lceil K \log n \rceil$ with K and θ large enough.

Proof. First we compare $W_{n',n}$ with $Z_{n'} R_{n',n}$. Lemma 3.3 entails that there are $c, \varepsilon_0 > 0$ and $\delta \in (0, 1)$ such that

$$\mathbb{E}[|Z_k - A_{k-1} Z_{k-1}|^s] \leq c \delta^k \lambda(s)^k, \quad s \in [\alpha, \alpha + \varepsilon_0].$$

Combining this with Lemma 5.1 and Lemma 5.4 we obtain that for $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} &\mathbb{P}[|W_{n',n} - Z_{n'} R_{n',n}| > t] \\ &\leq \sum_{k=n'+1}^n \mathbb{P}[|Z_k - A_{k-1} Z_{k-1}| (1 + R_{k,n}) > t (k - n')^{-2}/2] \\ &\leq c \sum_{k=n'+1}^n \mathbb{E}[|Z_k - A_{k-1} Z_{k-1}|^{\alpha+\varepsilon}] t^{-\alpha-\varepsilon} (k - n')^{2(\alpha+1)} \lambda(\alpha)^{n-k} (n - k)^{2(\alpha+1)} e^{(n-k)(\rho\varepsilon+c\varepsilon^2)} \\ &\leq ct^{-\varepsilon} t^{-\Lambda^*(\rho)/\rho} (\log n)^{4(\alpha+2)+1} n^{\theta(\rho\varepsilon+c\varepsilon^2)} \delta^{n'} = o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho}), \end{aligned}$$

where the last equality holds for sufficiently large θ . Next, recalling (3.25), we may write

$$Z_{n'} - Z_{\tilde{n}} \Pi_{\tilde{n},n'} = \sum_{k=\tilde{n}+1}^{n'} (Z_k - A_{k-1} Z_{k-1}) \Pi_{k,n'}.$$

By Lemma 5.4, for $\varepsilon = (\log t)^{-1/2}$ with t big enough

$$\mathbb{P}[|Z_{n'} - Z_{\tilde{n}} \Pi_{\tilde{n},n'}| R_{n',n} > t]$$

$$\begin{aligned}
 &\leq \sum_{k=\tilde{n}+1}^{n'} \mathbb{P}[|Z_k - A_{k-1}Z_{k-1}| \Pi_{k,n'} R_{n',n} > t (2(k - \tilde{n}))^{-2}] \\
 &\leq c \sum_{k=n'+1}^{n'} \mathbb{E}[|Z_k - A_{k-1}Z_{k-1}|^{\alpha+\varepsilon}] \lambda(\alpha + \varepsilon)^{n'-k} (\log n)^{2(\alpha+1)} \\
 &\quad \times t^{-\alpha-\varepsilon} (k - \tilde{n})^{2(\alpha+1)} e^{(n-n')(\Lambda(\alpha)+\rho\varepsilon+c\varepsilon^2)} \\
 &\leq ct^{-\Lambda^*(\rho)/\rho} e^{cn\varepsilon^2} \cdot e^{n'(\Lambda(\alpha+\varepsilon)-\Lambda(\alpha))} e^{-n'\varepsilon\rho} e^{-n'c\varepsilon^2} \cdot \sum_{k=\tilde{n}+1}^{n'} \delta^k (k - \tilde{n})^{2(\alpha+1)} \\
 &\leq ct^{-\Lambda^*(\rho)/\rho} \delta^{\tilde{n}} \leq C(\log t)^K \log \delta t^{-\Lambda^*(\rho)/\rho} = o((\log t)^{-1/2} t^{-\Lambda^*(\rho)/\rho}).
 \end{aligned}$$

for appropriately large K . □

Proof of Theorem 2.3. By previous considerations, we only need to consider

$$\mathbb{P}[Z_{\tilde{n}} \Pi_{\tilde{n},n'} R_{n',n} > t].$$

Denote $H_{n,j_n} = Z_{\tilde{n}} R_{n',n}$ and apply Lemma 3.6 to infer that

$$\mathbb{P}[Z_{\tilde{n}} \Pi_{\tilde{n},n'} R_{n',n} > t] = \mathbb{P}[H_{j_n} \Pi_{j_n,n} > t] \sim \frac{C_5(\alpha)}{\sqrt{\log t}} t^{-\Lambda^*(\rho)/\rho},$$

where $C_5(\alpha) = \frac{c_{RCZ}}{\alpha\sigma(\alpha)\sqrt{2\pi}}$.

The second claim of Theorem 2.3 is an immediate consequence of Lemma 5.5. □

5.2. Limit theorems of T_t^W

We start with the following lemma.

Lemma 5.8. *Suppose the assumptions of Theorem 2.1 and (1.3) are in force. Assume $\rho = \rho_0$ and let $n_1 = n - \lfloor b\sqrt{n \log n} \rfloor$ and $n_2 = n + \lfloor b\sqrt{n \log n} \rfloor$. Then for any $\delta > 0$ one can pick $b > 0$ large enough such that*

$$\mathbb{P}[W_{n_1} > t] \leq Ct^{-\alpha_0} (\log t)^{-\delta}$$

and

$$\mathbb{P}[W_{n_2,\infty} > t] \leq Ct^{-\alpha_0} (\log t)^{-\delta},$$

where $W_{n_2,\infty} = \sum_{j=n_2+1}^{\infty} Z_j$.

Proof. Applying Lemma 5.5 with $\varepsilon = \sqrt{\log n/n}$ we obtain

$$\begin{aligned}
 \mathbb{P}[W_{n_1} > t] &\leq Cn_1^{2(\alpha_0+1)} t^{-\alpha_0-\varepsilon} e^{n_1\rho\varepsilon+cn_1\varepsilon^2} \leq Ct^{-\alpha_0} n^{2(\alpha_0+1)} e^{-b\rho \log n} e^{c \log n} \\
 &\leq Ct^{-\alpha_0} n^{2(\alpha_0+1)+c-b} \leq Ct^{-\alpha_0} (\log t)^{-\delta}
 \end{aligned}$$

for appropriately large b .

To prove the second part of the lemma we proceed similarly as above, but this time ε depends also on the parameter k : $\varepsilon = \varepsilon(k, n) > 0$. We estimate

$$\begin{aligned}
 \mathbb{P}[W_{n_2,\infty} > t] &\leq \sum_{k=n_2+1}^{\infty} \mathbb{P}\left[Z_k > \frac{t}{2(k - n_2)^2}\right] \\
 &\leq C \sum_{k=n_2+1}^{\infty} \mathbb{E}[Z_k^{\alpha_0-\varepsilon(k,n)}] t^{-\alpha_0+\varepsilon(k,n)} (k - n_2)^{2\alpha_0}
 \end{aligned}$$

$$\leq Ct^{-\alpha_0} \sum_{k=n_2+1}^{\infty} t^{\varepsilon(k,n)} \lambda(\alpha_0 - \varepsilon(k,n))^k (k - n_2)^{2\alpha_0}.$$

Now for some large N we consider separately two cases when $k \leq Nn$ and $k > Nn$. First we consider large values of k for which we just choose $\varepsilon(k,n) = \varepsilon_2$ for some small fixed ε_2 . Let $\gamma = \lambda(\alpha_0 - \varepsilon_2) < 1$. Then

$$\begin{aligned} t^{-\alpha_0} \sum_{k=Nn}^{\infty} t^{\varepsilon(k,n)} \lambda(\alpha_0 - \varepsilon(k,n))^k (k - n_2)^{2\alpha_0} &\leq t^{-\alpha_0} t^{\varepsilon_2} \sum_{k>Nn} \gamma^k k^{2\alpha} \\ &\leq t^{-\alpha_0} t^{\varepsilon_2} \gamma^{\frac{nN}{2}} \sum_{k>Nn} \gamma^{k/2} k^{2\alpha} \leq Ct^{-\alpha_0} t^{-\delta} \end{aligned}$$

for appropriately large N . For small values of k choose $\varepsilon(k,n) = \varepsilon_1 = \sqrt{\log n/n}$ and recall that for some $c > 0$, $\lambda(\alpha_0 - \varepsilon) \leq e^{-\varepsilon\rho_0 + c\varepsilon^2}$. Then

$$\begin{aligned} t^{-\alpha_0} \sum_{k=n_2+1}^{Nn} t^{\varepsilon(k,n)} \lambda(\alpha_0 - \varepsilon(k,n))^k (k - n_2)^{2\alpha_0} &\leq t^{-\alpha_0} e^{\rho_0 \sqrt{n \log n}} \sum_{n_2 \leq k \leq Nn} e^{-\rho_0 k \sqrt{\log n/n}} e^{Ck \log n/n} (k - n_2)^{2\alpha_0} \\ &\leq t^{-\alpha_0} e^{\rho_0 \sqrt{n \log n}} \cdot Nn \cdot e^{-\rho_0 n_2 \sqrt{\log n/n}} e^{CN \log n} (Nn)^{2\alpha_0} \\ &\leq C_N t^{-\alpha_0} e^{-\rho_0 b \log n} n^{CN+2\alpha_0+1} \leq C_N t^{-\alpha_0} n^{-\rho_0 b + CN + 2\alpha_0 + 1} \\ &\leq C_N t^{-\alpha_0} (\log t)^{-\delta} \end{aligned}$$

by increasing b if necessary. □

Lemma 5.9. *For $\rho = \rho_0$ and $y \in \mathbb{R}$ we have*

$$\mathbb{P}[|W_{n_y} - Z_{\tilde{n}} \Pi_{\tilde{n}, n_1-1} R_{n_1, n_y-1}| > t] = o(t^{-\alpha_0}), \quad t \rightarrow \infty,$$

where $n_y = n_1 + \lfloor c_0 y \sqrt{\log n} \rfloor$, n_1 as in Lemma 5.8 and $\tilde{n} = \lceil K \log n \rceil$ for large K and $c_0 = \sigma_0 \rho_0^{-3/2}$.

This lemma can be proved exactly in the same way as Lemma 5.7. We left details for the reader.

Proof of Theorem 2.4. STEP 1. LAW OF LARGE NUMBERS. The weak law of large numbers is a direct consequence of Lemma 5.8 and (1.6). Indeed, for any $\varepsilon > 0$, we have

$$\begin{aligned} &\mathbb{P}\left[\left|\frac{T_t^W}{\log t} - \frac{1}{\rho_0}\right| > \varepsilon \mid T_t^W < \infty\right] \\ &\leq \mathbb{P}[T_t^W < \log t(1/\rho_0 - \varepsilon) \mid T_t^W < \infty] + \mathbb{P}[T_t^W > \log t(1/\rho_0 + \varepsilon) \mid T_t^W < \infty] \\ &\leq \mathbb{P}[W_{n_1} > t \mid T_t^W < \infty] + \mathbb{P}[W_{n_2, \infty} > t \mid T_t^W < \infty] \leq C(\log t)^{-\delta}, \end{aligned}$$

where the last but one inequality holds if $\sqrt{\log n/n} \leq \varepsilon$.

STEP 2. CENTRAL LIMIT THEOREM. The second part of the claim can be proved using similar arguments as in the proof of Theorem 2.2 in [11]. However in our case some additional problems arise. Thus we focus here on the main arguments, emphasising the differences. We refer the reader to [11] for all the details.

STEP 2A. PETROV'S RESULT. The result follows essentially from Petrov's Theorem (Lemma 3.4) and first we explain how it should be applied. In view of Lemma 5.9 we need prove that

$$\lim_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[Z_{n'} \Pi_{n', n_1-1} R_{n_1, n_y-1} > t] = C_{KG} c_Z(\alpha_0) \Phi(y),$$

where C_{KG} the Kesten–Goldie constant defined in (5.3) and $c_Z(\alpha_0)$ is the limiting constant in Lemma 3.1. We want to apply Lemma 3.4 conditionally on $Z_{n'} R_{n_1, n_y-1}$. This can be done only for some restricted set of values. The details are

as follows. Let $\rho_0 = \rho(\alpha_0)$ and $\sigma_0 = \sigma(\alpha_0)$ and

$$I(t) = [0, \rho_0(n - n_1 + n') + (y + \theta)\sigma_0\sqrt{n_1 - n'}],$$

$$V_n = Z_{n'}R_{n_1, n_y}.$$

We will focus on the event $\{V_n \in I(t)\}$ since we will later argue that its complement is negligible. Below we apply Lemma 3.4 with (n, t, δ_n) replaced by $(n_1 - n', e^{\rho_0(n_1 - n')}, \frac{\rho_0(n - n_1 + n') - s}{n_1 - n'})$. Let F_t be the distribution function of $\log V_n$ (recall that n depends on t), then

$$\begin{aligned} \mathbb{P}[V_n \Pi_{n', n_1} > t, V_n \in I(t)] &= \int_{I(t)} \mathbb{P}[\Pi_{n', n_1} > te^{-s}] dF_t(s) \\ &= \frac{1 + o(1)}{\alpha_0 \sigma_0 \sqrt{2\pi(n_1 - n')}} \int_{I(t)} t^{-\alpha_0} e^{\alpha_0 s} e^{-\frac{(n_1 - n')\delta_n^2}{2\sigma_0^2}} dF_t(s) \\ &= \frac{(1 + o(1))t^{-\alpha_0}}{\alpha_0 \sigma_0 \sqrt{2\pi(n_1 - n')}} \int_{I(t)} e^{\alpha_0 s} e^{-\frac{(\rho_0(n - n_1 + n') - s)^2}{2\sigma_0^2(n_1 - n')}} dF_t(s). \end{aligned}$$

Next we change variables applying the transformation

$$T_t(s) = \frac{s - \rho_0(n - n_1 + n')}{\sigma_0\sqrt{n_1 - n'}}$$

and defining the distribution $G_t = F_t \circ T_t^{-1}$ we obtain

$$(5.6) \quad \mathbb{P}[V_n \Pi_{n', n_y} > t, V_n \in I(t)] = \frac{(1 + o(1))t^{-\alpha_0}}{\alpha_0 \sigma_0 \sqrt{2\pi(n_1 - n')}} \int_{-\frac{\rho(n - n_1 + n')}{\sigma_0\sqrt{n_1 - n'}}}^{y + \theta} e^{\alpha_0 T_t^{-1}(u)} e^{-u^2/2} dG_t(u).$$

STEP 2B. UNIFORM CONVERGENCE. To proceed further let us define $\bar{F}_t(s) = \mathbb{P}[Z_{n'}R_{n_1, n_y} > e^s]$. We need the following technical observation that for $-\infty < a < b < y$

$$(5.7) \quad \lim_{t \rightarrow \infty} e^{\alpha_0 s} \bar{F}_t(s) = C_{\text{KGCZ}}(\alpha_0) \quad \text{uniformly for } s \in T_t^{-1}([a, b]).$$

To prove the above statement we apply (4.35) from [11]:

$$(5.8) \quad \lim_{t \rightarrow \infty} e^{\alpha_0 s} \mathbb{P}[R_{n_1, n_y} > e^s] \rightarrow C_{\text{KG}} \quad \text{uniformly for } s \in T_t^{-1}([a, b]).$$

Then

$$\begin{aligned} e^{\alpha_0 s} \bar{F}_t(s) &= e^{\alpha_0 s} \mathbb{P}[Z_{n'}R_{n_1, n_y} > e^s] \\ &= e^{\alpha_0 s} \sum_{k=1}^{\infty} \mathbb{P}[R_{n_1, n_y} > e^s/k] \mathbb{P}[Z_{n'} = k] \\ &= e^{\alpha_0 s} \sum_{k \leq e^{\delta\sqrt{n}}} \mathbb{P}[R_{n_1, n_y} > e^s/k] \mathbb{P}[Z_{n'} = k] + e^{\alpha_0 s} \sum_{k > e^{\delta\sqrt{n}}} \mathbb{P}[R_{n_1, n_y} > e^s/k] \mathbb{P}[Z_{n'} = k] \\ &= I + II. \end{aligned}$$

We will prove that the first term gives the asymptotic behaviour and the second one is negligible. To estimate the latter, by the Hölder inequality and (5.3), we may write

$$(5.9) \quad \begin{aligned} II &\leq C \sum_{k \geq e^{\delta\sqrt{n}}} k^{\alpha_0} \mathbb{P}[Z_{n'} = k] = \mathbb{E}[Z_{n'}^{\alpha_0} \mathbf{1}_{\{Z_{n'} > e^{\delta\sqrt{n}}\}}] \\ &\leq C \mathbb{E}[Z_{n'}^{p\alpha_0}]^{1/p} \mathbb{P}[Z_{n'} > e^{\delta\sqrt{n}}]^{1/q} \\ &\leq C e^{\Lambda(p\alpha_0)K \log n/p} e^{-\delta\alpha_0\sqrt{n}/q} \mathbb{E}[Z_{n'}^{\alpha_0}]^{1/q} = o(1). \end{aligned}$$

For $k \leq e^{\delta\sqrt{n}}$, $s - \log k \in T_t^{-1}([a - 2\delta, b])$ and by (5.8) we have that for small ε and large n

$$(C_{\text{KG}} - \varepsilon)\mathbb{E}[Z_{n'}^{\alpha_0} \mathbf{1}_{\{k \leq e^{\delta\sqrt{n}}\}}] \leq I \leq (C_{\text{KG}} + \varepsilon)\mathbb{E}[Z_{n'}^{\alpha_0} \mathbf{1}_{\{k \leq e^{\delta\sqrt{n}}\}}].$$

Applying (5.9), Lemma 3.1, passing first with $n \rightarrow \infty$ and then with $\varepsilon \rightarrow 0$, we obtain

$$\lim_{s \rightarrow \infty} I = C_{\text{KGC}}(\alpha_0).$$

STEP 2C. CONVERGENCE TO THE LEBESGUE MEASURE. Now our aim is to prove that for any $f \in C_C(-\infty, y)$ (continuous, compactly supported function in $(-\infty, y)$):

$$(5.10) \quad \lim_{t \rightarrow \infty} \int_{-\infty}^y f(u) dH_t(u) = C_{\text{KGC}}(\alpha_0) \int_{-\infty}^y f(u) du,$$

where $dH_t(u) = \frac{e^{\alpha_0 T_t^{-1}(u)}}{\alpha_0 \sigma_0 \sqrt{n_1 - n'}} dG_t(u)$ and

$$(5.11) \quad H_t(v, w) := \int_v^w dG_t(u) \leq C(w - v) + \frac{C}{\sqrt{n_1}}$$

for $-\infty < v < w < y$ and some constant C .

Fix $-\infty < v < w < y$, $v^*(t) = T_t^{-1}(v)$, $w^*(t) = T_t^{-1}(w)$, then integrating by parts

$$\begin{aligned} H_t(v, w) &= -\frac{1}{\alpha_0 \sigma_0 \sqrt{n_1 - n'}} (e^{\alpha_0 w^*(t)} \bar{F}_t(w^*(t)) - e^{\alpha_0 v^*(t)} \bar{F}_t(v^*(t))) + \frac{1}{\sigma_0 \sqrt{n_1 - n'}} \int_{v^*(t)}^{w^*(t)} e^{\alpha_0 u} \bar{F}_t(u) du. \end{aligned}$$

and (5.7) implies (5.11).

To prove (5.10) observe that the first term above is negligible and we again apply (5.7) and obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} H_t(v, w) &= \lim_{t \rightarrow \infty} \frac{1}{\sigma_0 \sqrt{n_1 - n'}} \int_{v^*(t)}^{w^*(t)} e^{\alpha_0 u} \bar{F}_t(u) du \\ &= \lim_{t \rightarrow \infty} \frac{C_{\text{KGC}}(\alpha_0)(w^*(t) - v^*(t))}{\sigma_0 \sqrt{n_1 - n'}} = C_{\text{KGC}}(\alpha_0)(w - v). \end{aligned}$$

Finally applying the standard procedure and approximating the integral by Riemann sums we obtain (5.10).

STEP 2D. CONCLUSION. Now we are able to conclude. For large N and small δ we split the integral (5.6) into three parts: $(-\frac{\rho_0(n-n_1+n')}{\sigma_0 \sqrt{n_1-n'}}, -N]$, $(-N, y - \delta]$, $(y - \delta, y)$, and in view of (5.10) and (5.11) we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{-N}^{y-\delta} e^{-u^2/2} dH_t(u) &= C_{\text{KGC}}(\alpha_0) \int_{-N}^{y-\delta} e^{-u^2/2} du, \\ \lim_{t \rightarrow \infty} \int_{-\frac{\rho_0(n-n_1+n')}{\sigma_0 \sqrt{n_1-n'}}}^{-N} e^{-u^2/2} dH_t(u) &\leq e^{-N^2/2}, \\ \lim_{t \rightarrow \infty} \int_{y-\delta}^y e^{-u^2/2} dH_t(u) &\leq C\delta. \end{aligned}$$

Passing with $N \rightarrow \infty$ and $\delta \rightarrow 0$ we obtain

$$\lim_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[V_n \Pi_{n', n_y} > t, V_n \in I(t)] = \frac{C_{\text{KGC}}(\alpha_0)}{\sqrt{2\pi}} \int_{-\infty}^y e^{-u^2/2} du = \frac{C_{\text{KGC}}(\alpha_0)}{\sqrt{2\pi}} \Phi(y)$$

STEP 2E. THE NEGLIGIBLE PART. To complete the proof we need to justify that the remaining part is negligible, i.e.

$$\lim_{t \rightarrow \infty} t^{\alpha_0} \mathbb{P}[V_n \Pi_{n', n_y} > t, V_n \notin I(t)] = 0.$$

However we omit the arguments here and refer to [11] (proof of Theorem 2, step 4) for more details. □

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